UNIT IV – MAGNETOSTATICS

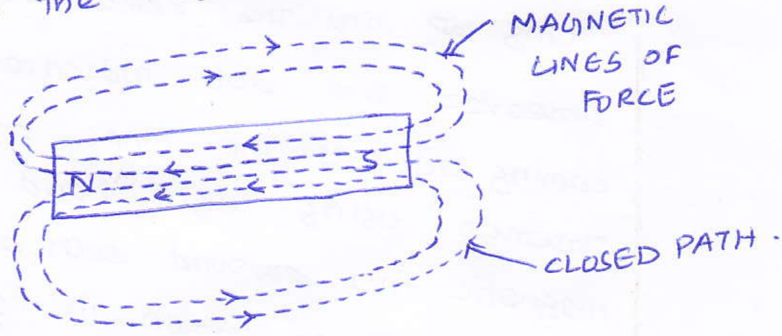
MAGNETIC FIELD & ITS PROPERTIES.

Consider a Permanent magnet, It has two Poles North (N) and South (S). The region around a magnet within which the influence of the magnet can be experienced is called magnetic field.

Such a field is represented by imaginary lines around the magnet which are called magnetic lines of force. These lines of force are also called magnetic lines of flux or magnetic flux lines.

An important difference b/w electric flux lines and magnetic flux lines can be observed here. In case of electric flux, the flux lines originate from an isolated Positive charge and diverge to terminate at infinity. while for a -ve charge, electric flux lines converge on a charge, starting from infinity. But in case of magnetic flux, the Poles exists in Pairs only.

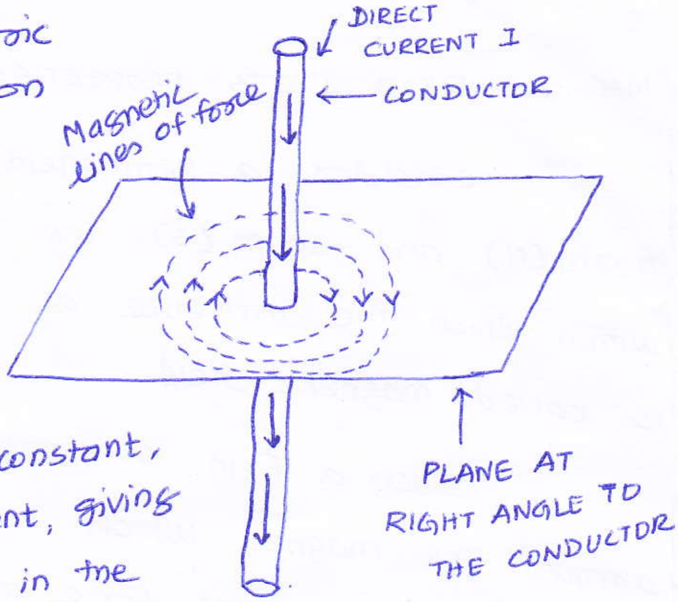
Hence every magnetic flux line starting from North Pole must end at south Pole and complete the path from south to north internal to the magnet.



MAGNETIC FIELD DUE TO CURRENT CARRYING CONDUCTOR:

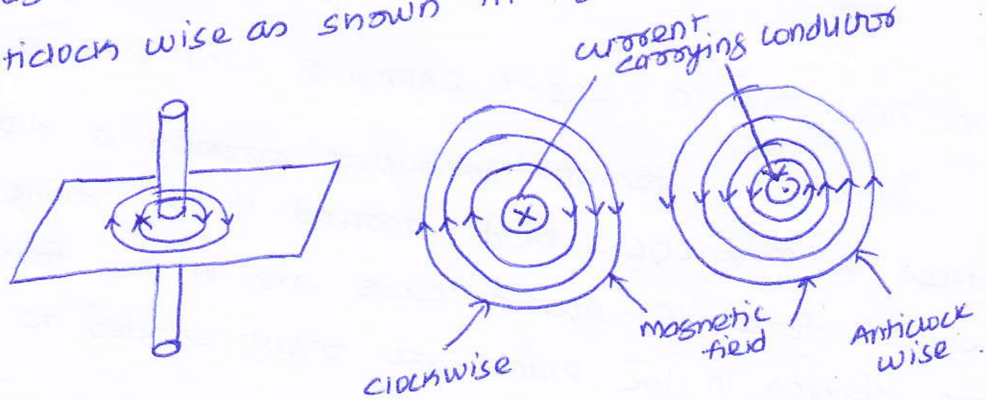
When a straight conductor carries a direct current, it Produces a magnetic field around it, all along its length. The lines of force in such a case are in the form of concentric circles in the planes at right angles to the conductor.

The direction of concentric circles around depends on the direction of current thro' the conductor. As long as the current is constant and current is time independent, magnetic lines of force are also constant, static and time independent, giving a steady magnetic field in the space around the conductor.



A right hand thumb rule is used to determine the direction of magnetic field around a conductor carrying a direct current. It states that, hold the current carrying conductor in right hand such that thumb pointing in the direction of current and ||el to the conductor, then curled fingers point in the direction of the magnetic lines of flux around it.

The cross indicates that the current direction is going into the plane of the paper away from the observer. The dot indicates that the current direction is coming out of the plane of the paper, coming towards the observer. Using right hand thumb rule, the direction of magnetic flux around such a conductor is either clockwise (or) anticlockwise as shown in figure.



The magnetic lines of force i.e magnetic flux lines always form a closed loop and exist in the form of concentric circles, around a current carrying conductor. The total number of magnetic lines of force is called a magnetic flux denoted as ϕ. It is measured in weber (Wb). one Wb means 10^8 lines of force.

MAGNETIC FIELD INTENSITY : ($\vec{H}$).

The quantitative measure of strongness or weakness of the magnetic field is given by magnetic field intensity or magnetic field strength. The magnetic field intensity at any point in the magnetic ~~field~~ field is defined as the force experienced by a unit north pole of one weber strength, when placed at that point.

The magnetic flux lines are measured in webers (Wb) while magnetic field intensity is measured in Netwons/weber [N/Wb] or amperes per meter [A/m]. or ampere turns/meter [AT/m]. It is denoted by $\vec{H}$.

MAGNETIC FLUX DENSITY : ($\vec{B}$)

The Total magnetic lines of force i.e magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density. It is denoted by $\vec{B}$. It is measured in Wb/m^2 which is also called Tesla (T).

RELATION BETWEEN $\vec{B}$ & $\vec{H}$.

In magnetostatics, the $\vec{B}$ and $\vec{H}$ are related to each other thro' the property of the region in which current carrying conductor is placed. It is called Permeability denoted as μ.

For free space

Permability is denoted as $\mu_0 = 4\pi \times 10^{-7}$ H/m

For any other region

a relative Permability is specified as μ_r

and $\mu = \mu_0 \mu_r$.

The $\vec{B}$ and $\vec{H}$ are related as

$$\boxed{\vec{B} = \mu \vec{H} = \mu_0 \mu_r \vec{H}}$$

For free space

$$\boxed{\vec{B} = \mu_0 \vec{H}}$$

For nonmagnetic media

$\mu_r = 0$

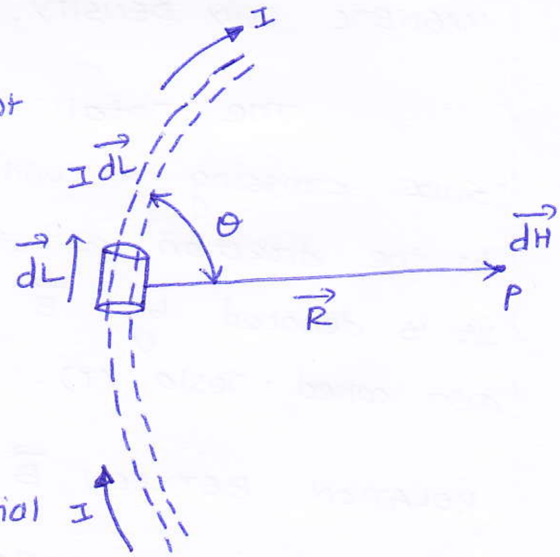
For magnetic materials

$\mu_r > 1$.

BIOT-SAVART LAW:

Consider a conductor carrying a direct current I and a steady magnetic field Produced around it. The Biot-savart law allows us to obtain the differential magnetic field intensity $d\vec{H}$, Produced at point P, due to a differential current element I dL.

consider a differential length dL hence the differential current element is IdL. This is a very small Part of the current carrying conductor. The Point P is at a distance R from the differential current element. The θ is the angle b/w the differential current element and the line joining Point P to the differential current element.



The Biot-Savarts law states that

The magnetic field intensity $\vec{dH}$ produced at a point P due to a differential current element IdL is,

1. Proportional to the product of current I and differential length dL.

2. The sine of the angle b/w the element and the line joining point P to the element

3. And inversely proportional to the square of the distance R b/w point P and the element.

Mathematically, the Biot-Savarts law can be stated as,

$$\vec{dH} \propto \frac{IdL \sin\theta}{R^2} \quad \text{---} ①$$

$$\vec{dH} = \frac{KIdL \sin\theta}{R^2} \quad \text{----} ②$$

$K \to$ constant of proportionality

$$K = \frac{1}{4\pi}$$

$$\therefore \vec{dH} = \frac{IdL \sin\theta}{4\pi R^2} \quad \text{-----} ③$$

Let dL = Magnitude of vector length $\vec{dL}$ and

$\vec{a_R}$ = unit vector in the direction from differential current element to point P.

Then from rule of cross product

$$\vec{dL} \times \vec{a_R} = dL\, |\vec{a_R}| \sin\theta = dL \sin\theta \quad \text{---} ④$$

using ④ in ③ we get

$$\vec{dH} = \frac{I\, \vec{dL} \times \vec{a_R}}{4\pi R^2} \text{ A/m} \quad \text{---} ⑤$$

But $\vec{a_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$ $\therefore$ $\boxed{\vec{dH} = \frac{I\, \vec{dL} \times \vec{R}}{4\pi R^3} \text{ A/m}}$ --- ⑥

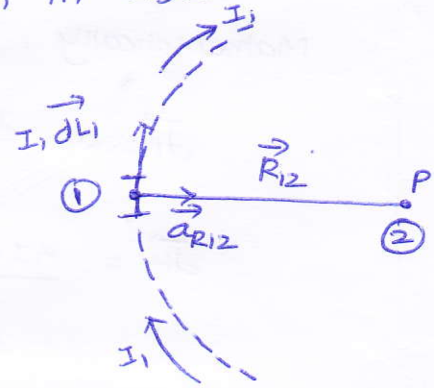
Equations ⑤ and ⑥ is the mathematical form of Biot-savarts law.

The entire conductor is made up of all such differential elements. Hence to obtain total magnetic field Intensity $\vec{H}$, the equation ⑤ takes the integral form as,

$$\vec{H} = \oint \frac{I\,\vec{dL} \times \vec{a_R}}{4\pi R^2} \quad ---- ⑦$$

The closed line integral is required to ensure that all the current elements are considered. This is because current can flow only in the closed path, provided by the closed circuit.

If the current element is considered at Point 1 and point P at Point 2, as shown in figure then,



$$\vec{dH_2} = \frac{I_1\,\vec{dL_1} \times \vec{a_{R12}}}{4\pi R_{12}^2} \text{ A/m.}$$

$I_1 \rightarrow$ current flowing thro' dL_1 at Point 1

$\vec{dL_1} \rightarrow$ Differential vector length at Point 1

$\vec{a_{R12}} \rightarrow$ unit vector in the direction from element at Point 1 to the Point P at Point 2

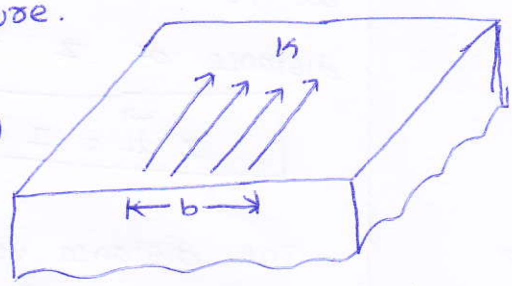
$$\vec{a_{R12}} = \frac{\vec{R_{12}}}{|\vec{R_{12}}|} = \frac{\vec{R_{12}}}{R_{12}}$$

$$\boxed{\therefore\ H = \oint \frac{I_1 \vec{dL_1} \times \vec{a_{R12}}}{4\pi R_{12}^2} \text{ A/m.}} \quad --- ⑧$$

This is called Integral form of Biot-Savart Law.

BIOT-SAVART LAW INTERMS OF DISTRIBUTED SOURCES

Consider a surface carrying a uniform current over its surface as shown in figure. Then the surface current density is denoted as $\vec{K}$ and is measured in (A/m)



Thus for uniform current density, current I in any width b is given by $I = Kb$, where width b is $\perp^r$ to the direction of current flow.

Thus if ds is the differential surface area considered of a surface having current density $\vec{K}$ then

$$I\,\vec{dL} = \vec{K}\,\vec{ds} \quad ---- (1)$$

If the current density in a volume of a given conductor is $\vec{J}$ measured in A/m^2 Then the differential volume dv we can write

$$I\,\vec{dL} = \vec{J}\,dv \quad --- (2)$$

Hence biot-savart's law can be expressed for surface current considering $\vec{K}$ ds while for volume current considering $\vec{J}$ dv.

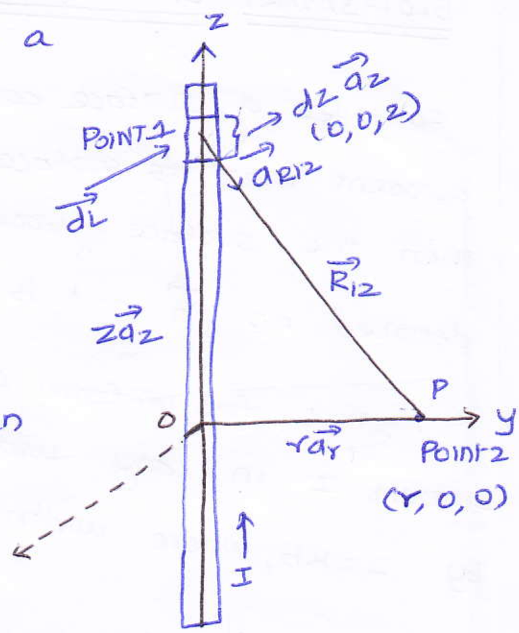
$$\therefore\ \boxed{\begin{aligned} \vec{H} &= \oint_S \frac{\vec{K} \times \vec{a}_R\, ds}{4\pi R^2}\ A/m \\ \vec{H} &= \oint_{vol} \frac{\vec{J} \times \vec{a}_R\, dv}{4\pi R^2}\ A/m \end{aligned}} \quad ---- (3)$$

The Biot-savart's law is also called Ampere's law for the current element.

$\vec{H}$ due to infinitely Long straight conductor:

Consider an infinitely long straight conductor, The conductor is along z-axis The current passing through The conductor is a direct current of I Amp. The field intensity $\vec{H}$ at a point P is to be calculated, which is at a distance 'r' from the z-axis.

Consider small differential element at point 1, along the z-axis at a distance of z from origin.



$$\therefore \boxed{I\vec{dL} = I\,dz\,\vec{a_z}} \quad \text{--- (1)}$$

The distance vector joining point 1 to point 2 is $\vec{R_{12}}$ and can be written as

$$\vec{R_{12}} = (r-0)\vec{a_r} + (0-0)\vec{a_\phi} + (0-z)\vec{a_z}$$

$$= r\vec{a_r} + 0\vec{a_\phi} - z\vec{a_z}$$

$$\therefore \boxed{\vec{R_{12}} = r\vec{a_r} - z\vec{a_z}} \quad \text{--- (2)}$$

$$\vec{a}_{R12} = \frac{\vec{R_{12}}}{|\vec{R_{12}}|} = \frac{-z\vec{a_z} + r\vec{a_r}}{\sqrt{r^2+z^2}} = \frac{r\vec{a_r} - z\vec{a_z}}{\sqrt{r^2+z^2}}$$

$$\vec{dL} \times \vec{a}_{R12} = \begin{bmatrix} \vec{a_r} & \vec{a_\phi} & \vec{a_z} \\ 0 & 0 & dz \\ r & 0 & -z \end{bmatrix} = r\,dz\,\vec{a_\phi}$$

Note: while taking cross product, $|R_{12}|$ is neglected for convenience and must be considered for further calculations.

$$\therefore I\vec{dL} \times \vec{a}_{R12} = \frac{I r\,dz\,\vec{a_\phi}}{\sqrt{r^2+z^2}}$$

According to Biot-Savart's law, $\vec{dH}$ at point 2 is

$$\vec{dH} = \frac{I\vec{dL} \times \vec{a}_{R12}}{4\pi R_{12}^2} = \frac{I r\,dz\,\vec{a_\phi}}{\sqrt{r^2+z^2}} \cdot \frac{1}{4\pi\left(\sqrt{r^2+z^2}\right)^2}$$

$$\frac{I\vec{dL} \times \vec{a}_{R12}}{4\pi R^2} =$$

$$\frac{I\,dz\,\vec{a_z}}{4\pi(r^2+z^2)} \times \frac{r\vec{a_r} - z\vec{a_z}}{\sqrt{r^2+z^2}}$$

$$\vec{dH} = \frac{I r\,dz\,\vec{a_\phi}}{4\pi(r^2+z^2)^{3/2}}.$$

$a_z \times a_r = a_\phi \qquad a_z \times a_z = 0$

$$\boxed{\vec{dH} = \frac{I r\,dz}{4\pi(r^2+z^2)^{3/2}}\,a_\phi}$$

Thus total flux intensity $\vec{H}$ can be obtained by integrating $\vec{dH}$ over the entire length of the conductor.

$$\therefore \vec{H} = \int_{z=-\infty}^{\infty} \vec{dH} = \int_{z=-\infty}^{\infty} \frac{I r dz \, \vec{a_\phi}}{4\pi (r^2+z^2)^{3/2}}$$

Put $z = r\tan\theta$ ← can be obtained by using $z = r\tan\theta$.

$z^2 = r^2\tan^2\theta$ $\quad z=-\infty,\ \theta = -\pi/2$ & $z = +\infty,\ \theta = +\pi/2$

and $dz = r\sec^2\theta \, d\theta$,

$$\therefore \vec{H} = \int_{\theta=-\pi/2}^{\theta=\pi/2} \frac{I r \, r\sec^2\theta \, d\theta \, \vec{a_\phi}}{4\pi (r^2+r^2\tan^2\theta)^{3/2}}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{I r^2 \sec^2\theta \, d\theta \, \vec{a_\phi}}{4\pi r^3 (\sec^3\theta)}$$

$\left[\because (r^2+r^2\tan^2\theta)^{3/2} = \left[r^2(1+\tan^2\theta)\right]^{3/2} = r^3\sec^3\theta\right]$ $\quad 1+\tan^2\theta = \sec^2\theta$.

$$\vec{H} = \int_{-\pi/2}^{\pi/2} \frac{I}{4\pi r} \cdot \frac{1}{\sec\theta} \cdot d\theta \cdot \vec{a_\phi} \Rightarrow \frac{I}{4\pi r}\int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta \, \vec{a_\phi}$$

$$\vec{H} = \frac{I}{4\pi r}\left[\sin\theta\right]_{-\pi/2}^{\pi/2} \vec{a_\phi} = \frac{I}{4\pi r}\left[\sin \pi/2 - \sin(-\pi/2)\right]\vec{a_\phi}$$

$$\vec{H} = \frac{I}{4\pi r}\left[1-(-1)\right]\vec{a_\phi} = \frac{2I}{4\pi r}\vec{a_\phi}$$

$$\boxed{\vec{H} = \frac{I}{2\pi r}\vec{a_\phi}} \text{ A/m} \qquad \boxed{\vec{B} = \mu\vec{H} = \frac{\mu I}{2\pi r}\vec{a_\phi} \text{ Wb/m}^2}$$

AMPERE'S CIRCUITAL LAW:

In electrostatics, the Gauss's law is useful to obtain the $\vec{E}$ in case of complex problems. Similarly in the magneto-statics, the complex problems can be solved using a law called Ampere's circuital law (or) Ampere's work law.

The ampere's circuital law states that,

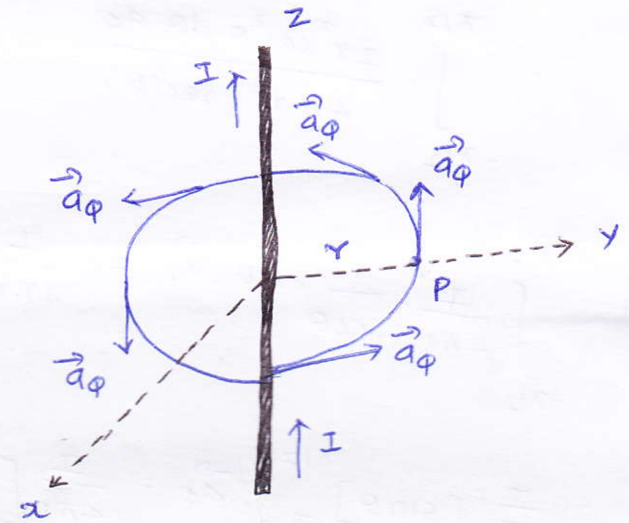
The line integral of magnetic field intensity $\vec{H}$ around a closed path is exactly equal to the direct current enclosed by that path.

The mathematical representation of Ampere's circuital law is,

$$\boxed{\oint \vec{H} \cdot \vec{dL} = I}$$

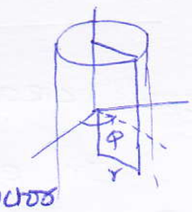
PROOF FOR AMPERE'S CIRCUITAL LAW:

Consider a long straight conductor carrying direct current I placed along z-axis as shown in figure below.



Consider a closed circular path of radius r which encloses the straight conductor carrying direct current I. The point P is at a $\perp^r$ distance r from the conductor. consider $\vec{dL}$ at point P which is in $\vec{a}_\varphi$ direction, tangential to circular path at point P.

$$\therefore \vec{dL} = r\, d\varphi\, \vec{a}_\varphi \quad \text{---} (1)$$



while $\vec{H}$ obtained at point P, from Biot-savart's law due to infinitely long conductor is

$$\vec{H} = \frac{I}{2\pi r} \vec{a}_\varphi \quad \text{[from Page NO (5) of unit IV]} \quad \text{--} (2)$$

$$\therefore \vec{H}.\vec{dL} = \frac{I}{2\pi r}\vec{a_\phi} \cdot r\,d\phi\, a_\phi$$

$$\vec{H}.\vec{dL} = \frac{I r\, d\phi}{2\pi r}$$

$$\vec{H}.\vec{dL} = \frac{I d\phi}{2\pi} \quad --- (3)$$

Integrating (3) over the entire closed Path,

$$\oint \vec{H}.\vec{dL} = \int_{\phi=0}^{2\pi} \frac{I\,d\phi}{2\pi} = \frac{I}{2\pi}\left[\phi\right]_0^{2\pi} = I$$

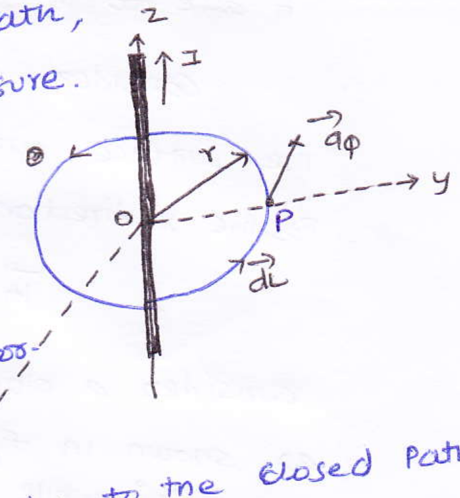
= current carried by conductor.

This Proves that the integral $\vec{H}.\vec{dL}$ along closed path gives direct current enclosed by that closed Path.

APPLICATION OF AMPERE'S CIRCUITAL LAW:

$\vec{H}$ due to infinitely straight conductor

Consider an infinitely long straight ~~line~~ conductor Placed along z-axis, carrying a direct current I as shown in the figure. consider the Amperian closed Path, enclosing the conductor as shown in figure. consider Point P on the closed path at which $\vec{H}$ is to be obtained. The radius of the Path is r and hence P is at a $\perp^r$ distance from the conductor.



The magnitude of $\vec{H}$ depends on r and the direction is always tangential to the closed path i.e $\vec{a_\phi}$. So $\vec{H}$ has only component in $\vec{a_\phi}$ direction say H_ϕ.

Consider elementary length $\vec{dL}$ at a point P and in cylindrical co-ordinates it is $r\,d\phi$ in $\vec{a_\phi}$ direction.

$$\therefore \vec{H} = H_\phi \vec{a_\phi} \quad \text{and} \quad \vec{dL} = r\,d\phi\,\vec{a_\phi}$$

$$\therefore \vec{H}.\vec{dL} = H_\phi \vec{a_\phi} \cdot r\,d\phi\,\vec{a_\phi} = H_\phi\, r\,d\phi$$

According to Ampere's circuital law,

$$\oint \vec{H} \cdot \vec{dL} = I.$$

$$\int_{\phi=0}^{2\pi} H_\phi r \, d\phi = I$$

$$H_\phi r \left[\phi\right]_0^{2\pi} = I$$

$$H_\phi r \, 2\pi = I$$

$$\Rightarrow \quad H_\phi = \frac{I}{2\pi r}$$

Hence $\vec{H}$ at Point P is given by

$$\boxed{\vec{H} = H_\phi \vec{a_\phi} = \frac{I}{2\pi r} \vec{a_\phi} \quad A/m}$$

$$\therefore \boxed{\vec{B} = \mu \vec{H} = \frac{\mu I}{2\pi r} \vec{a_\phi}}$$

$\vec{H}$ due to infinite sheet of current:

Consider an infinite sheet of current in the $z=0$ plane. The surface current density is $\vec{K}$. The current is flowing in Positive y direction hence

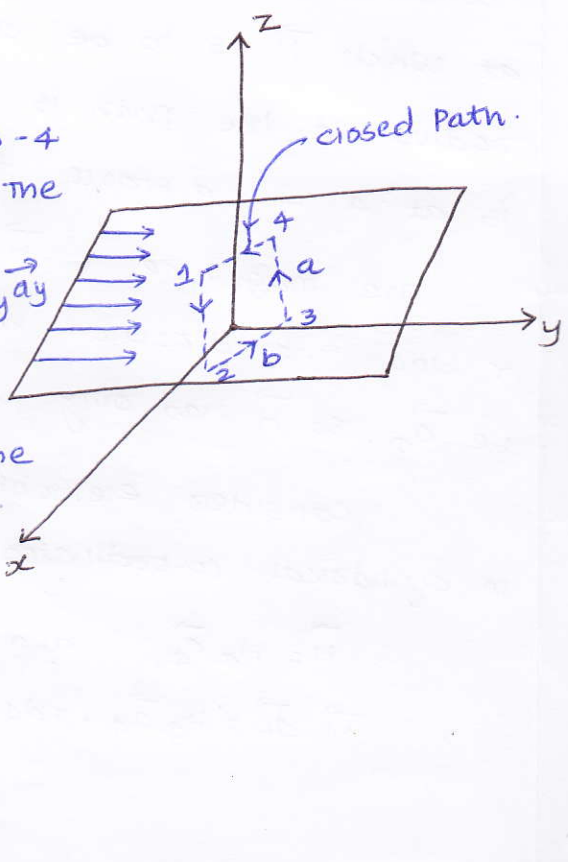
$$\vec{K} = K_y \vec{a_y}$$

Consider a closed path 1-2-3-4 as shown in fig. The width of the Path is 'b' while height is 'a'.

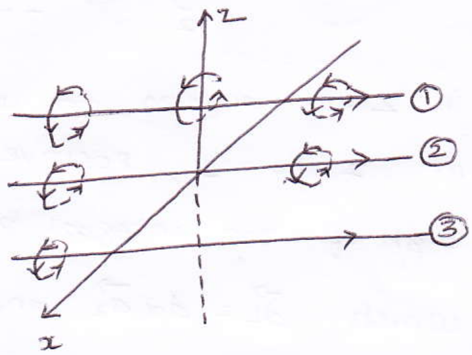
It is $\perp^r$ to the direction of current hence in xz plane.

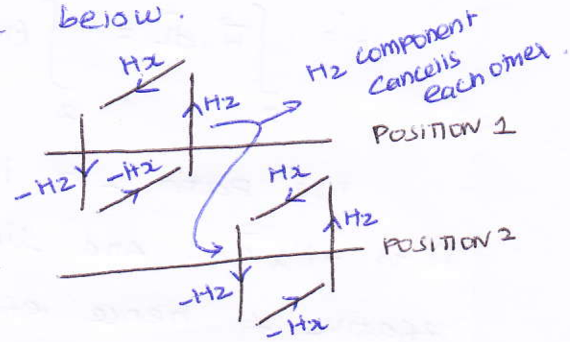
The current flowing across the distance 'b' is given by $K_y b$

$$\therefore I_{enc} = K_y b \quad \text{---- (1)}$$



consider the magnetic lines of force due to the current in $\vec{a_y}$ direction, according to the right hand thumb rule. These are shown in figure below.





It is clear that in b/w two very closely spaced conductors, the components of $\vec{H}$ in z direction are oppositely directed [$-H_z$ for position 1 and $+H_z$ for position 2 b/w the two opposite positions]. All such components cancel eachother and hence $\vec{H}$ cannot have any component in z-direction.

As current is flowing in y direction, $\vec{H}$ cannot have component in y direction.

so $\vec{H}$ has only component in x direction.

$$\left.\begin{aligned}\vec{H} &= H_x \vec{a_x} \quad \text{for } z>0 \\ &= -H_x \vec{a_x} \quad \text{for } z<0\end{aligned}\right\} \quad \text{---- (2)}$$

Applying Ampere's circuital law

$$\oint \vec{H}\cdot\vec{dL} = I_{enc} \quad \text{---- (3)} = \left[\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1 \vec{H}\cdot\vec{dL}\right]$$

Evaluate the integral along the path 1-2-3-4-1.

For path 1-2, $\vec{dL} = dz\,\vec{a_z}$

3-4, $\vec{dL} = -dz\,\vec{a_z}$

But $\vec{H}$ is in x-direction while $\vec{a_x}\cdot\vec{a_z} = 0$.

Hence along the paths 1-2 and 3-4, the integral

$$\oint \vec{H}\cdot\vec{dL} = 0.$$

consider path 2-3 along which $\vec{dL} = dx\,\vec{a_x}$

$$\therefore \int_2^3 \vec{H}\cdot\vec{dL} = \int_2^3 (-H_x\vec{a_x})\cdot(dx\,\vec{a_x}) = -H_x\int_2^3 dx = bH_x \quad \text{--- (4)}$$

The path 2-3 is lying in $z<0$ region for which $\vec{H}$ is $-H_x\vec{a_x}$. And limits from 2 to 3, positive x to negative x hence effective sign of the integral is +ve

consider path 4-1 along which $\vec{dL} = dx\,\vec{a_x}$ and its in the region $z>0$ hence $\vec{H} = H_x\vec{a_x}$

$$\therefore \int_4^1 \vec{H}\cdot\vec{dL} = \int_4^1 (H_x\vec{a_x})\cdot(dx\,\vec{a_x}) = H_x\int_4^1 dx = bH_x \quad \text{--- (5)}$$

$\therefore$ sub (4) and (5) in (3) we get

$$\oint \vec{H}\cdot\vec{dL} = bH_x + bH_x = 2bH_x \quad \text{--- (6)}$$

usin (1) in eqn (6) we get

$$\oint \vec{H}\cdot\vec{dL} = 2bH_x = I_{enc} = K_y b$$

$$\therefore 2bH_x = K_y b$$

$$H_x = \frac{1}{2}K_y$$

Hence $\vec{H} = \frac{1}{2}K_y\vec{a_x}$ for $z>0$

$= -\frac{1}{2}K_y\vec{a_x}$ for $z<0$

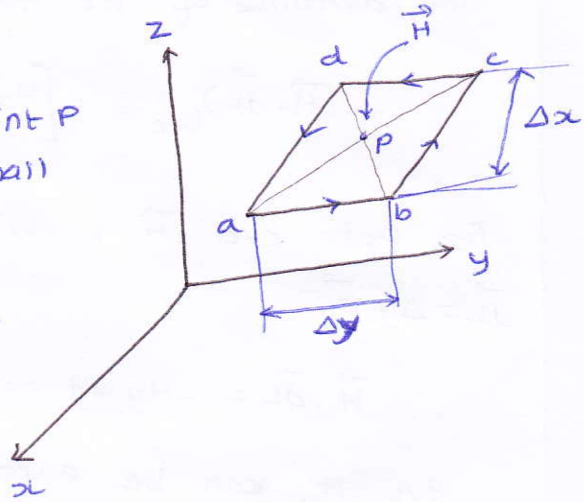
In general for an infinite sheet of current density $\vec{K}$ A/m we can write

$$\vec{H} = \frac{1}{2}\vec{K}\times\vec{a_N}$$

where $\vec{a_N}$ = unit vector normal from the current sheet to the point at which $\vec{H}$ is to be obtained.

CURL:

Consider the differential surface element having sides Δx and Δy plane, as shown in figure below. The unknown current has produced $\vec{H}$ at the centre of the incremental closed path.



The total magnetic field at point P which is at the centre of the small rectangle is,

$$\vec{H} = H_{x0}\vec{a_x} + H_{y0}\vec{a_y} + H_{z0}\vec{a_z} \quad --- (1)$$

while the total current density is given by,

$$\vec{J} = J_x\vec{a_x} + J_y\vec{a_y} + J_z\vec{a_z} \quad --- (2)$$

To apply Ampere's circuital law to this closed path, let us evaluate the closed line integral of $\vec{H}$ about this path in the direction abcda. According to right hand thumb rule the current is in $\vec{a_z}$ direction.

Along path a-b,

$$\vec{H} = H_y\vec{a_y} \ \& \ \vec{dL} = \Delta y\,\vec{a_y}$$

$$\therefore \vec{H}.\vec{dL} = H_y\,\Delta y \quad ---- (3)$$

The intensity H_y along a-b can be expressed interms of H_{y0} existing at P and the rate of change of H_y in the x-direction with x.

The distance in x direction of a-b from point P is $\left(\frac{\Delta x}{2}\right)$. Hence $\vec{H}.\vec{dL}$ along a-b can be expressed as.

$$(\vec{H}.\vec{dL})_{ab} = \left[H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{\Delta x}{2}\right)\right]\Delta y \quad --- (4)$$

For path b-c, $\vec{H}$ is in $-\vec{a_x}$ direction hence $-H_x\vec{a_x}$ and $\vec{dL} = \Delta x\,\vec{a_x}$

$$\therefore \vec{H}.\vec{dL} = -H_x\,\Delta x \quad --- (5)$$

Now H_x can be expressed interms of H_{x0} at point P and rate of change of H_x in y direction and y.

$$H_x = H_{x0} + \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}$$

The distance of bc from P is $\Delta y/2$.

$$\therefore (\vec{H}\cdot\vec{dL})_{b-c} = -\left[H_{x0} + \frac{\partial H_x}{\partial y}\frac{\Delta y}{2}\right]\Delta x \quad --- (6)$$

For path c-d $\vec{H}$ is in $-\vec{a_y}$ direction hence $-H_y\vec{a_y}$ and $\vec{dL} = \Delta y\,\vec{a_y}$

$$\therefore \vec{H}\cdot\vec{dL} = -H_y\,\Delta y \quad ---- (7)$$

But H_y can be expressed interms of H_{y0} and rate of change of H_y in negative x direction. The distance of CD from point P is $(\Delta x/2)$ in negative x direction

$$\therefore H_y = H_{y0} - \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}$$

$$\therefore (\vec{H}\cdot\vec{dL})_{c-d} = -\left[H_{y0} - \frac{\Delta x}{2}\frac{\partial H_y}{\partial x}\right]\Delta y. \quad --- (8)$$

For path d-a, $\vec{H}$ is in $+\vec{a_x}$ direction hence $H_x\vec{a_x}$ and $\vec{dL} = \Delta x\,\vec{a_x}$

$$\therefore \vec{H}\cdot\vec{dL} = H_x\,\Delta x \quad --- (9)$$

But H_x can be expressed interms of H_{x0} and rate of change of H_x in negative y direction. The distance of DA from point P is $(\frac{\Delta y}{2})$ in negative y direction

$$\therefore H_x = \left[H_{x0} - \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}\right]$$

$$\therefore (\vec{H}\cdot\vec{dL})_{d-a} = \left[H_{x0} - \frac{\Delta y}{2}\frac{\partial H_x}{\partial y}\right]\Delta x \quad --- (10)$$

Total $\vec{H}\cdot\vec{dL}$ can be obtained by adding equations (4), (6), (8) and (10)

$$\therefore \vec{H}\cdot\vec{dL} = H_{y0}\Delta y + \frac{\Delta x \Delta y}{2}\frac{\partial H_y}{\partial x} - H_{x0}\Delta x - \frac{\Delta x \Delta y}{2}\frac{\partial H_x}{\partial y}$$

$$- H_{y0}\Delta y + \frac{\Delta x \Delta y}{2}\frac{\partial H_y}{\partial x} + H_{x0}\Delta x - \frac{\Delta x \Delta y}{2}\frac{\partial H_x}{\partial y}.$$

$$\therefore \oint \vec{H}\cdot\vec{dL} = \Delta x \Delta y\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] \text{ ----- } (11)$$

According to Ampere's circuital law, this integral must be current enclosed by the differential element.

current enclosed = $\left.\begin{array}{l}\text{current density normal to}\\ \text{closed path}\end{array}\right\} \times$ Area of that closed path

$$I_{encl.} = J_z \Delta x \Delta y \text{ --- } (12)$$

where J_z = current density in $\vec{a_z}$ direction as the current enclosed is in $\vec{a_z}$ direction.

From eqn (11) & (12)

$$\oint \vec{H}\cdot\vec{dL} = I_{encl.}$$

$$\oint \vec{H}\cdot\vec{dL} = \Delta x \Delta y\left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] = J_z \Delta x \Delta y$$

$\div\ \Delta x \Delta y$

$$\frac{\oint \vec{H}\cdot\vec{dL}}{\Delta x \Delta y} = \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right] = J_z \text{ --- } (13)$$

This gives accurate result as the closed path shrinks to a point (ie) $\Delta x \Delta y$ area tends to zero.

$$\therefore \lim_{\Delta x \Delta y \to 0} \frac{\oint \vec{H}\cdot\vec{dL}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \text{ --- } (14)$$

considering incremental closed path in yz plane we get the current density normal to it i.e in y-direction. so we can write,

$$\lim_{\Delta y \Delta z \to 0} \frac{\oint \vec{H} \cdot \vec{dL}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \quad \text{--- (15)}$$

and

$$\lim_{\Delta z \Delta x \to 0} \frac{\oint \vec{H} \cdot \vec{dL}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \quad \text{--- (16)}$$

In general we can write,

curl.

$$\boxed{\lim_{\Delta S_N \to 0} \frac{\oint \vec{H} \cdot \vec{dL}}{\Delta S_N}} = J_N \quad \text{--- (17)}$$

where J_N = current density normal to the surface Δs.

The term on left hand side of the equation is called the closed line integral. curl $\vec{H}$. The ΔS_N is area enclosed by the total $\vec{J}$ now can be obtained by adding eqn (14), (15) and (16).

$$\therefore \vec{J} = J_x \vec{a_x} + J_y \vec{a_y} + J_z \vec{a_z}$$

$$= \left[\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right]\vec{a_x} + \left[\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right]\vec{a_y} + \left[\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right]\vec{a_z}$$

$$\boxed{\vec{J} = \text{curl } \vec{H} = \nabla \times \vec{H}} \quad \text{--- (18)}$$

The curl $\vec{H}$ is indicated by $\nabla \times \vec{H}$ which is cross product of operator 'del' and $\vec{H}$

The equation (18) is called the point form of Ampere's circuital law.

$$\therefore \boxed{\text{curl } \vec{H} = \nabla \times \vec{H} = \vec{J}}$$

This is one of the Maxwell's equations.

The curl of a vector in the direction of the unit vector is the ratio of the line integral of the vector around a closed contour, to the enclosed area bounded by the contour, as the enclosed area diminishes to zero.

Properties of curl:

1. The curl of a vector is a vector quantity
2. $\nabla \times (\vec{A}+\vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$
3. $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 A$
4. The divergence of a curl is zero
   $\nabla \cdot (\nabla \times \vec{A}) = 0.$
5. The curl of a gradient of a vector is zero
   $\nabla \times \nabla V = 0.$

STROKE'S THEOREM:

Analogous to the divergence theorem in electrostatics, there exist stroke's theorem in magnetostatics. The stroke's theorem relates the line integral to a surface integral.

The stroke's theorem states that,

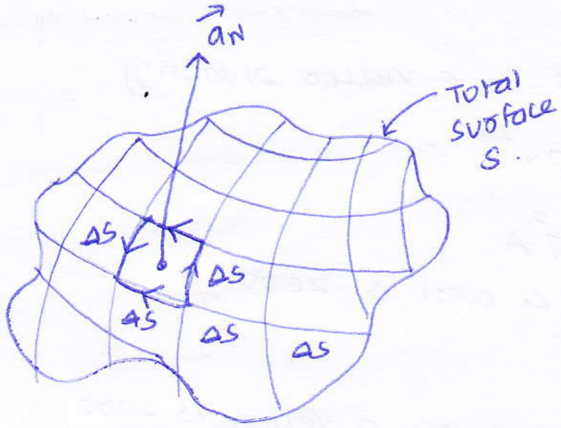
"The line integral of vector $\vec{A}$ around a closed path L is equal to the integral of curl of $\vec{A}$ over the open surface S enclosed by the closed path L".

The theorem is applicable only when $\vec{A}$ and $\nabla \times \vec{A}$ are continuous on the surface S.

$$\boxed{\oint_L \vec{H} \cdot \vec{dL} = \int_S (\nabla \times \vec{H}) \cdot \vec{ds}}$$

PROOF'S OF STROKE'S THEOREM:

Consider a surface S which is splitted into number of incremental surfaces. Each incremental surface is having area Δs as shown in Figure.



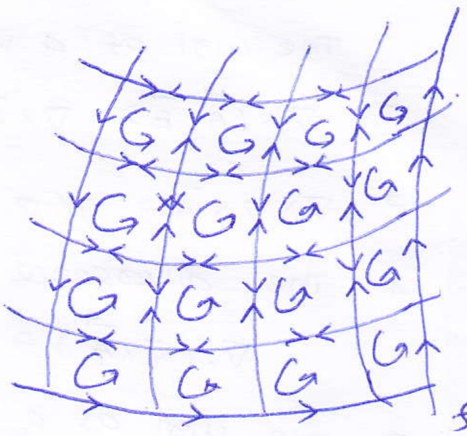


fig (b)

Applying by definition of the curl to any of these incremental surfaces we can write

$$(\nabla \times \vec{H})_N = \frac{\oint \vec{H} \cdot \vec{dL}_{\Delta s}}{\Delta s} \quad --- (1)$$

where
$N \rightarrow$ Normal to Δs according to right hand rule

$dL_{\Delta s} \rightarrow$ Perimeter of the incremental surface Δs.

Now the curl of $\vec{H}$ in the normal direction is the dot Product of curl of $\vec{H}$ with $\vec{a_N}$, where $\vec{a_N}$ is unit vector, normal to the surface Δs, according to right hand rule,

$$\therefore (\nabla \times \vec{H})_N = (\nabla \times \vec{H}) \cdot \vec{a_N}$$

$$\therefore \oint \vec{H} \cdot \vec{dL}_{\Delta s} = (\nabla \times \vec{H}) \cdot \vec{a_N} \, \Delta s.$$

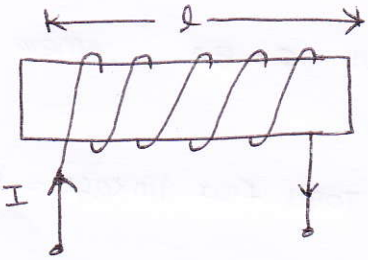
$$\therefore \oint \vec{H} \cdot \vec{dL}_{\Delta s} = (\nabla \times \vec{H}) \cdot \vec{\Delta s}$$

To obtain total curl for every incremental surface, add the closed line integrals for each Δs. From fig (b), it can be seen that at a common boundary b/w the two incremental surfaces, the line integral is getting cancelled as the boundary is getting traced in two opposite directions.

INDUCTANCE OF A SOLENOID:

Consider a solenoid of N turns as shown in fig.

Let the current flowing thro' the solenoid be I Amps. Let the length of the solenoid be l and the cross-section area be A.



Magnetic field intensity H inside the solenoid is given by,

$$H = \frac{NI}{l} \,(A/m) \quad \text{---} \; (1)$$

Total flux linkage is given by

Total flux linkage $= N\phi = N(B)(A)$

$$\left[\because \phi = BA \quad Wb = \frac{Wb}{m^2} \times m^2\right]$$

$$N\phi = NBA$$

But $B = \mu H$

$$\therefore N\phi = N(\mu H)A$$

$$= N\mu HA$$

$$= N\mu\left[\frac{NI}{l}\right]A \Rightarrow N\phi = \frac{\mu N^2 IA}{l}$$

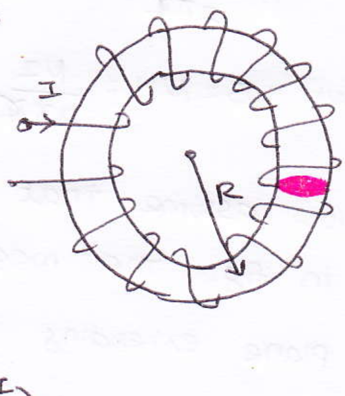
Inductance of a solenoid is given by

$$L = \frac{\text{Total flux linkage}}{\text{Total current}} = \frac{\mu N^2 IA/l}{I}$$

$$\boxed{L = \frac{\mu N^2 A}{l}}$$

INDUCTANCE OF A TOROID:

Consider a toroid ring with N turns and carrying current I. Let the radius of the toroid be R as shown in fig.



The magnetic flux density inside a toroidal ring is given by

$$B = \frac{\mu NI}{2\pi R} \quad \text{---} \; (1)$$

$$\left[\because B = \mu H = \mu \cdot \left(\frac{NI}{l}\right) \right.$$ → Perimeter of circle

$$= \mu\left(\frac{NI}{2\pi R}\right)$$

Total flux of a toroidal ring having N turns is given by,

Total flux linkage $= N\phi$

But $\phi = BA$ where $A \rightarrow$ Area of cross-section of a toroidal ring.

$\therefore$ Total flux linkage $= N(B)(A) = N\left[\frac{\mu NI}{2\pi R}\right](A) = \frac{\mu N^2 IA}{(2\pi R)}$

The inductance of a toroid is given by,

$$L = \frac{\text{Total flux linkage}}{\text{Total current}}$$

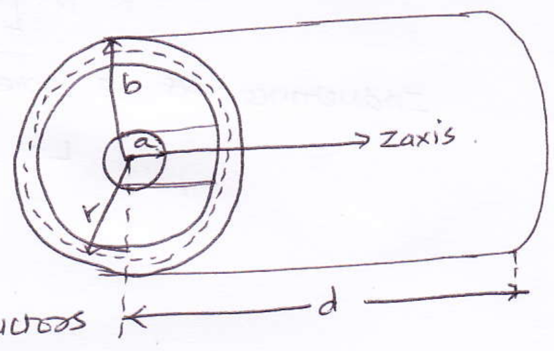
$$= \frac{\mu N^2 IA}{(2\pi R) I} = \frac{\mu N^2 A}{2\pi R} \; H.$$

$$\boxed{L = \frac{\mu N^2 A}{2\pi R} \; H}$$

Where $A \Rightarrow$ Area of cross section of Toroidal ring $= \pi r^2 \; m^2$

INDUCTANCE OF A CO-AXIAL CABLE:

Consider a co-axial cable with inner conductor radius 'a' and outer conductor radius 'b'.



Let the current thro' the co-axial cable be I.

For the co-axial cable the magnetic field intensity at any point b/w inner and outer conductors is given by

$$H = \frac{I}{2\pi r} \quad \text{where} \quad a < r < b \; \text{---} \; (1)$$

But $B = \mu H = \frac{\mu I}{\ldots}$ --- (2)

Let $\vec{B} = \frac{NI}{2\pi r} \vec{a_\phi}$ (T)

The Total magnetic flux is given by

$$\phi = \int_S \vec{B} \cdot \vec{ds}$$

Now $\vec{ds} = dr\,dz\,\vec{a_\phi}$ [from cylindrical co-ordinate system].

$$\therefore \phi = \int_{z=0}^{z=d} \int_{r=a}^{r=b} \frac{NI}{2\pi r} \vec{a_\phi} \cdot dr\,dz\,\vec{a_\phi}$$

$$= \frac{NI}{2\pi} \int_{z=0}^{z=d} dz \int_{r=a}^{r=b} \frac{1}{r}\,dr$$

$$= \frac{NI}{2\pi} [z]_0^d \, [\ln r]_a^b$$

$$\boxed{\phi = \frac{NI}{2\pi} d \ln\left(\frac{b}{a}\right)} \quad --- \ 3$$

The inductance of a co-axial cable is given by

$$L = \frac{\text{Total flux linkage}}{\text{Total current}}$$

$$\boxed{L = \frac{\frac{NId}{2\pi} \ln\left(\frac{b}{a}\right)}{I} = \frac{Nd}{2\pi} \ln\left(\frac{b}{a}\right) \text{ H}.}$$

The Inductance of a co-axial cable may be expressed per unit length as

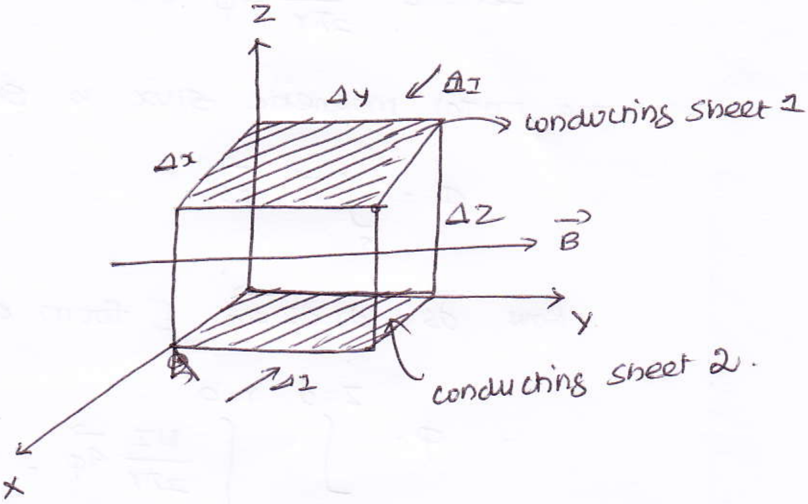
$$\boxed{\frac{L}{d} = \frac{N}{2\pi} \ln\left(\frac{b}{a}\right) \text{ H/m}.}$$

MAGNETIC ENERGY - ENERGY STORED IN A MAGNETIC FIELD

Energy stored in a inductor is given by

$$W_m = \frac{1}{2} L I^2$$

$$L = \frac{\Phi}{I} = \cancel{B}$$



$$\Delta L = \frac{\Delta \Phi}{\Delta I} = \frac{B\,\Delta s}{\Delta I}$$

Δs = differential surface area

$$= \Delta x\, \Delta z$$

$$\therefore \Delta L = \frac{B(\Delta x \Delta z)}{\Delta I} \quad \text{--- (1)}$$

$$B = \mu H$$

$$\therefore \Delta L = \frac{\mu H \Delta x \Delta z}{\Delta I}$$

$$\Delta I = H(\Delta y) \quad \text{--- (2)}$$

($\because$ conducting sheet current is in y direction).

$\therefore$ Energy stored

$$\Delta W_m = \frac{1}{2}\Delta L\, \Delta I^2 \quad \text{--- (3)}$$

sub (1) & (2) in (3) we get

$$\Delta W_m = \frac{1}{2}\left[\frac{\mu H \Delta x \Delta z}{\Delta I}\right]\left[H(\Delta y)\right]^2$$

$$= \frac{1}{2}\mu H^2(\Delta x \Delta y \Delta z)$$

But $\Delta V = \Delta x \Delta y \Delta z$

$$\therefore \Delta W_m = \frac{1}{2}\mu H^2 \Delta V$$

magnetostatics energy density function

$$W_m = \lim_{\Delta V \to 0} \frac{\Delta W_m}{\Delta V} = \frac{1}{2}\mu H^2 \quad (J/m^3)$$

different forms

$$W_m = \frac{1}{2}(\mu H)H = \frac{1}{2}BH$$

$$W_m = \frac{1}{2}\mu\left(\frac{B}{\mu}\right) = \frac{B^2}{}$$