UNIT-III

CONDUCTORS, DIELECTRICS AND CAPACITORS

CURRENT:

- Flow of charges constitutes an Electric current
- It can be measured by measuring how many charges are passing thro' a specified surface or a point in a material per second.
- It is rate of flow of charge at a specified point or across a specified surface per unit time is called an **Electric current**.
- It is measured in Ampere, which is coulombs/sec (c/s).

i.e. $\boxed{I = \frac{dQ}{dt} \; c/s \text{ i.e Amps}}$

A current of 1 Amp is said to be flowing across the surface when a charge of one coulomb is passing across the surface in one second.

CURRENT DENSITY:

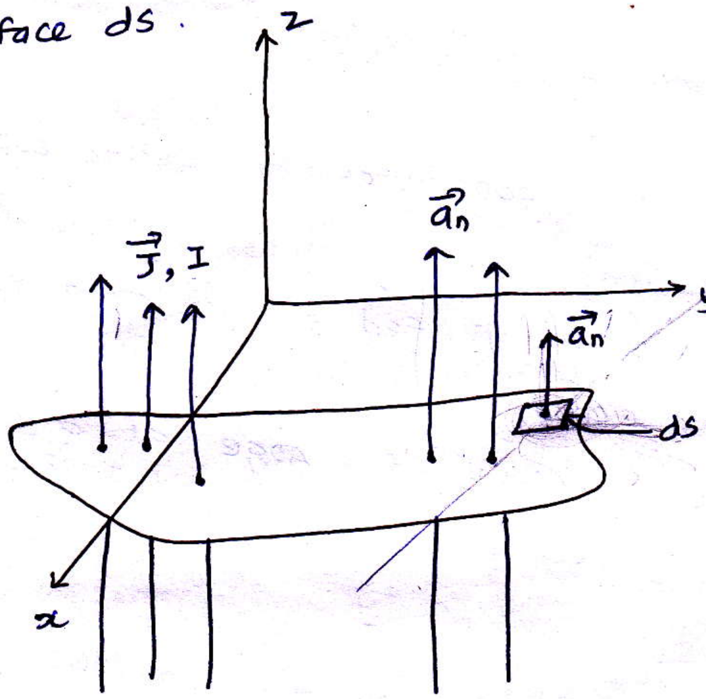
It is defined as the current passing thro' the unit surface area, when the surface is held normal to direction of the current.

- It is a vector quantity and denoted as $\vec{J}$.
- It is measured in Amperes per sq.meters (A/m^2).

RELATIONSHIP BETWEEN I AND J:

Consider a surface S and I is the current passing thro' the surface. The direction of current is normal to the surface S and hence direction of $\vec{J}$ is also normal to the surface S.

consider an incremental area ds as shown in fig below and $\vec{a_n}$ is the unit vector normal to the incremental surface ds.



A/m^2 m^2

$I = J \times S$

$dI = \vec{J} \cdot \vec{ds}$

$\vec{ds} = ds\,\vec{a_n}$

$\vec{J} = J\,\vec{a_n}$

Then the differential current dI passing thro' the differential surface ds is given by the dot product of the current density vector $\vec{J}$ and $\vec{ds}$.

$\therefore \quad dI = \vec{J} \cdot \vec{ds}$ [dot product]

When $\vec{J}$ and $\vec{ds}$ are at right angles ($\theta = 0°$) then

$dI = \vec{J} \cdot \vec{ds} = |\vec{J}|\,|\vec{ds}| \cos 0°$

$$\boxed{dI = J\,ds}$$

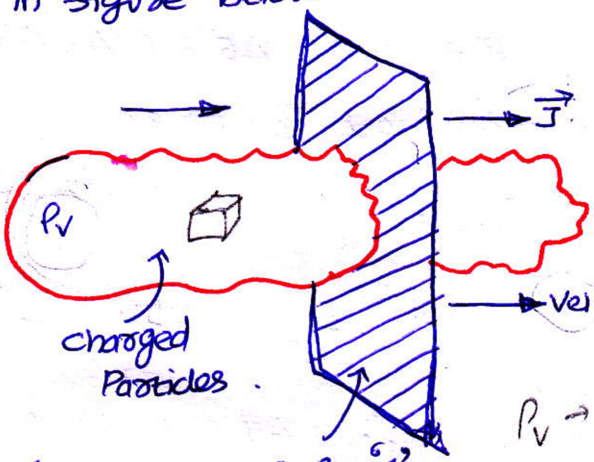
and $I = \oint_S J\,ds$. $J \to$ current density in A/m^2

But if $\vec{J}$ is not normal to $\vec{ds}$ then the total current is obtained by integrating $\vec{J} \cdot \vec{ds}$

$$\boxed{I = \oint_S \vec{J} \cdot \vec{ds}}$$

RELATION BETWEEN $\vec{J}$ & ρ_v

The set of charged particles give rise to a charge density ρ_v in a volume V. The current density $\vec{J}$ can be related to the velocity with the volume charge density. i.e charged particles in volume V crosses the surface S at a point. This is shown in figure below



The velocity with which the charge is getting transferred is $\vec{v}$ m/s. This is a vector quantity.

$\rho_v \to C/m^3 \times m^3$

To derive the relation between $\vec{J}$ and ρ_v, consider differential volume ΔV having charge density ρ_v as show in figure below. The elementary charge that volume carries is,

$Q = \rho_v V$

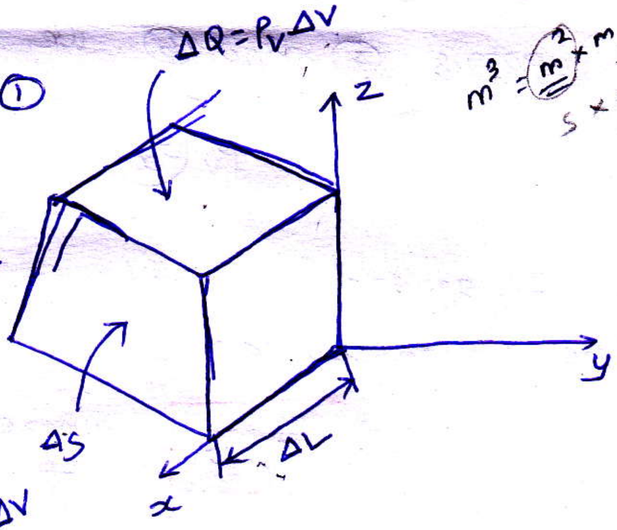
$\Delta Q = \rho_v \Delta V$

m^3

$$\boxed{\Delta Q = \rho_v \Delta V} \quad \text{--- (1)}$$

$m^3 = m^2 \times m$

$S \times L$

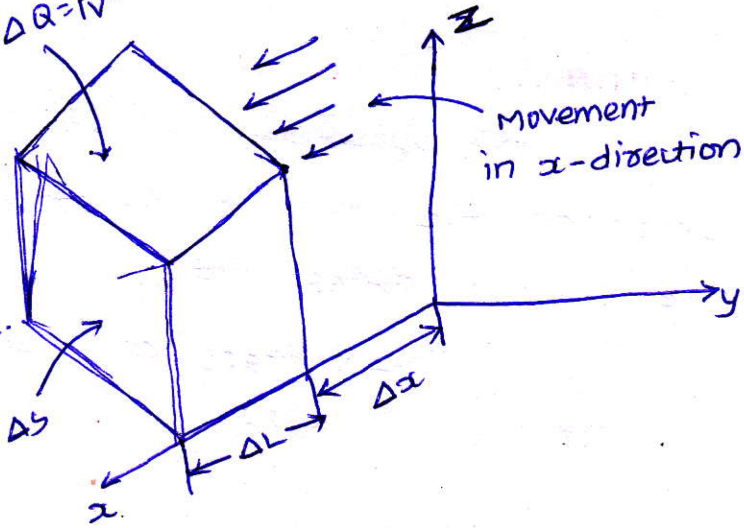


Let ΔL is the incremental length while ΔS is the incremental surface area & incremental volume is,

$$\boxed{\Delta V = \Delta S \, \Delta L} \quad \text{--- (2)}$$

$\therefore$ sub (2) in (1) we get

$$\boxed{\Delta Q = \rho_v \Delta S \, \Delta L} \quad \text{--- (3)}$$



Let the charge is moving in x-direction with velocity $\vec{v}$ and thus velocity has only x component v_x.

In the time interval Δt the element of charge has moved through distance Δx, in x direction. The direction is normal to the surface ΔS and hence the resultant current can be expressed as,

$$\boxed{\Delta I = \frac{\Delta Q}{\Delta t} \quad --- (4)}$$

But now, $\Delta Q = \rho_v \Delta S \Delta x$ as the charge corresponding the length Δx is moved and responsible for the current.

$$\therefore \quad \Delta I = \rho_v \Delta S \frac{\Delta x}{\Delta t} \quad ---- (5)$$

But $\frac{\Delta x}{\Delta t}$ = velocity in x-direction i.e. v_x

$$\therefore \quad \Delta I = \rho_v \Delta S \underbrace{v_x}_{\text{X component of velocity } \vec{v}} \quad --- (6)$$

But $\Delta I = \vec{J} \Delta S$ --- (7) when $\vec{J}$ and ΔS are normal.

Here $\vec{J}$ and ΔS are normal to each other hence comparing (7) and (6) we get

$$J_x = \rho_v \, v_x = x \text{ component of } \vec{J} \quad ----- (8)$$

In general, the relation ship between $\vec{J}$ and ρ_v can be expressed as

$$\boxed{\vec{J} = \rho_v \vec{v}}$$

where $\vec{v}$ is the velocity vector.

CONTINUITY EQUATION:

The continuity equation of the current is based on the principle of conservation of charge. The principle states that.

The charges can neither be created nor be destroyed.

consider a closed surface S with a current density $\vec{J}$, then the total current I crossing the surface S is given by,

$$I = \oint_S \vec{J} \cdot \vec{ds} \quad --- (1)$$

- The current flows outwards from the closed surface.
- The current means the flow of positive charges.
- The current I is constituted due to outward flow of +ve charge from the closed surface S.

- According to principle of conservation of charge, there must be decrease of an equal amount of +ve charge inside the closed surface.

- Hence the outward rate of flow of +ve charge gets balanced by the rate of decrease of charge inside the closed surface.

Let Q_i = charge within the closed surface

$-\frac{dQ_i}{dt}$ = Rate of decrease of charge inside the closed surface.

The Negative sign indicates decrease in charge.

Due to principle of conservation of charge, this rate of decrease is same as rate of outward flow of charge, which is current

$$\therefore \quad I = \oint_S \vec{J} \cdot \vec{ds} = -\frac{dQ_i}{dt} \quad --- (2) \quad [\text{outward flowing current } I]$$

This is the integral form of the continuity equation of the current.

Current entering the volume is

$$\oint_S \vec{J} \cdot \vec{ds} = -I = \frac{dQ_i}{dt} \quad --- (3)$$

The point form of the continuity equation can be obtained from the integral form.

Using divergence theorem, convert the surface integral in integral form to the volume integral.

$$\left[\oint_S \vec{D}\cdot\vec{ds} = \int_{Vol}(\nabla\cdot\vec{D})dv\right] \quad \because \oint_S \vec{J}\cdot\vec{ds} = \int_{Vol}(\nabla\cdot\vec{J})dv \quad \text{---- (4)}$$

$$\therefore \quad -\frac{dQ_i}{dt} = \int_{Vol}(\nabla\cdot\vec{J})dv$$

But $Q_i = \int_{Vol} \rho_v\, dv$ where $\rho_v \rightarrow$ volume charge density.

$$\therefore \int_{Vol}(\nabla\cdot\vec{J})dv = -\frac{d}{dt}\left[\int_{Vol}\rho_v\, dv\right] = -\int_{Vol}\frac{d\rho_v}{\partial t}dv.$$

for constant surface derivative becomes partial derivative

$$\therefore \int_{Vol}(\nabla\cdot\vec{J})dv = -\int_{Vol}\frac{\partial\rho_v}{\partial t}dv. \quad \text{--- (5)}$$

If the above relation is true for any volume, it must be true for incremental volume Δv

$$\therefore (\nabla\cdot\vec{J})\Delta v = -\frac{\partial\rho_v}{\partial t}\Delta v$$

$$\therefore \boxed{(\nabla\cdot\vec{J}) = -\frac{\partial\rho_v}{\partial t}} \quad \text{--- (6)}$$

→ Point form or differential form of continuity equation of the current.

For steady currents which are not the function of time $\frac{\partial\rho_v}{\partial t} = 0$ hence

$(\nabla\cdot\vec{J}) = 0$. for steady state.

CONDUCTORS :

under the effect of applied electric field, The avaliable free electrons starts moving. The moving electrons strikes the adjacent atoms and rebound in the random directions. this is called drifting of electrons.

After some time, The electrons attain the constant average velocity called drift velocity (v_d). The current constituted due to the drifting of such electrons in metallic conductors is called drift current.

The drift velocity is directly Proportional to the applied electric field.

$$\vec{v}_d \propto \vec{E} \quad \text{---} \quad (1)$$

The constant of Proportionality is called mobility of the electrons in a given material and denoted as μ_e.

$$\vec{v}_d = -\mu_e \vec{E} \quad \text{---} \quad (2)$$

-ve sign indicates the velocity of the electrons is against the direction of field $\vec{E}$. [μ = 0.0012 for Al
= 0.0032 for Cu].

According to relation between $\vec{J}$ and $\vec{v}$ we can write,

$$\vec{J} = \rho_v \vec{v} \quad \text{---} \quad (3)$$

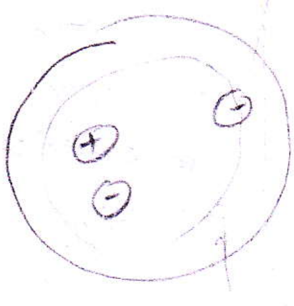
The drift velocity is the velocity of free electrons hence the above relation can be expressed as

$$\vec{J} = \rho_e \vec{v}_d \quad \text{---} \quad (4)$$

Where $\rho_e \rightarrow$ charge density due to free Electrons.

sub (2) in (4) we get

$$\boxed{\vec{J} = -\rho_e \mu_e \vec{E}} \quad \text{--} \quad (5)$$



POINT FORM OF OHM'S LAW:

The relationship between $\vec{J}$ and $\vec{E}$ can also be expressed in terms of conductivity of the material.

Thus for a metallic conductor

$$\boxed{\vec{J} = \sigma \vec{E}} \quad \text{---} \; (6)$$

σ - sigma

Where σ = conductivity of the material. ($\mho/m$) (mho/metre)

Point form of ohm's law.

$\sigma = 3.82 \times 10^7$ for Al

$= 5.8 \times 10^7$ for Cu.

By comparing (5) and (6)

$$\boxed{\sigma = -\rho_e \, \mu_e} \quad \text{----} \; (7)$$

$\rho_e = ne$

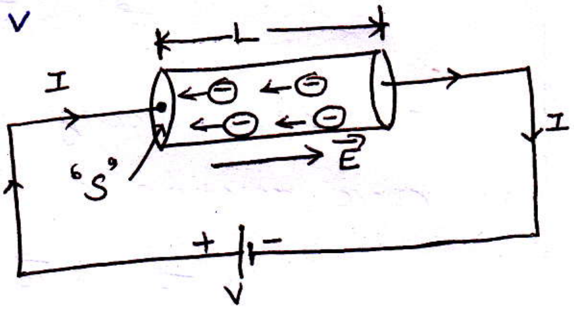
n - no. of electrons

e - charge per electron.

The resistivity is the reciprocal of the conductivity. The conductivity depends on the temperature. As temperature increases, the conductivity decreases and resistivity increases.

RESISTANCE OF A CONDUCTOR:

Consider that the voltage V is applied to a conductor of length L having uniform cross section 'S' as shown in the figure.



The direction of $\vec{E}$ is same as the direction of conventional current, which is opposite to flow of electrons. The electric field applied is uniform and its magnitude is given by,

$$E = \frac{V}{L} \quad \text{----} \; (1)$$

The conductor has uniform cross-section S and hence we can write

$$I = \int_S \vec{J} \cdot \vec{dS} = JS \quad \text{---} \; (2)$$

The current direction is normal to the surface dS.

Thus $J = \frac{I}{S} = \sigma E$ --- (3)

$J = \sigma E$

sub (1) in (3) we get

$E = V/L$

$E = V/L$

$J = \frac{\sigma V}{L}$ where σ = conductivity of the material

$J = \frac{\sigma V}{L}$

$V = \frac{JL}{\sigma} = \frac{IL}{\sigma S}$ $\left[\because J = \frac{I}{S}\right]$

$V = \frac{JL}{\sigma}$ $\left(J = \frac{I}{S}\right)$

$V = \frac{IL}{\sigma S}$

$V = \left(\frac{L}{\sigma S}\right) I$

$V = \left(\frac{L}{\sigma S}\right) I$

$$\boxed{\therefore R = \frac{V}{I} = \frac{L}{\sigma S}}$$

$R = \frac{V}{I}$

For nonuniform fields, the resistance R is defined as the ratio V to I where V is the potential difference b/w two specified equipotential surfaces in the material and I is the current crossing the more positive surface of the two, into the material.

$$\therefore R = \frac{V_{ab}}{I} = \frac{-\int_b^a \vec{E}\cdot\vec{dL}}{\int_s \vec{J}\cdot\vec{ds}} = \frac{-\int_b^a \vec{E}\cdot\vec{dL}}{\int_s \sigma\vec{E}\cdot\vec{ds}}$$

PROPERTIES OF CONDUCTOR:

1. Under static conditions, no charge and no electric field can exist at any point within the conducting material.

2. The charge can exist on the surface of the conductor giving rise to surface charge density

3. Within a conductor, the charge density is always zero

4. The charge distribution on the surface depends on the shape of the surface.

5. The conductivity of ideal conductor is infinite

6. The conductor surface is an equipotential surface.

Properties of Dielectric materials.

1. The dielectrics donot contain any free charges but contain bound charges

2. Bound charges are under the internal molecular and atomic forces and cannot contribute to the conduction.

3. When subjected to an external field $\vec{E}$, the bound charges shift their relative positions. Due to this, small electric dipoles get induced inside the dielectric. This is called Polarization.

4. Due to the Polarization, the dielectrics can store the energy

5. Due to the Polarization, the flux density of the dielectric increases by amount equal to the Polarization.

6. The induced dipoles produce their own electric field and align in the direction of the applied electric field.

7. When Polarization occurs, the volume charge density is formed inside the dielectric while the surface charge density is formed over the surface of the dielectric

8. The electric field outside and inside the dielectric gets modified due to the induced electric dipoles.

RELAXATION TIME:

The medium is called homogeneous when the physical characteristics of the medium donot vary from point to point but remain same everywhere throughout the medium.

If the characteristics vary from point to point, the medium is called hetrogeneous (or) non-homogeneous.

While the medium is called linear with respect to the electric field if the flux density $\vec{D}$ is diretly proportional to the electric field $\vec{E}$. The relationship is thro' the permittivity of the medium

If $\vec{D}$ is not directly proportional to $\vec{E}$, the material is called non-linear

Considering a conducting material which is linear and homogeneous. The current density for such a material is,

$$\vec{J} = \sigma \vec{E} \quad \text{where } \sigma \rightarrow \text{conductivity} \quad \text{---} (1)$$

But W.K.T $\vec{D} = \varepsilon \vec{E}$ --- (2)

$$\therefore \vec{E} = \frac{\vec{D}}{\varepsilon} \quad \text{---} (3)$$

sub (3) in (1) we get

$$\vec{J} = \sigma \frac{\vec{D}}{\varepsilon} = \frac{\sigma}{\varepsilon}\vec{D} \quad \text{---} (4)$$

The point form of continuity equation states that,

$$\nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} \quad \text{---} (5)$$

sub (4) in (5)

$$\nabla \cdot \left(\frac{\sigma}{\varepsilon}\vec{D}\right) = -\frac{\partial \rho_v}{\partial t}$$

$$\frac{\sigma}{\varepsilon}\nabla \cdot \vec{D} = -\frac{\partial \rho_v}{\partial t} \quad \text{---} (6)$$

But W.K.T $\nabla \cdot \vec{D} = \rho_v$ --- (7) $\rightarrow$ Point form of Gauss's law Page No: 18 (unit II)

$\therefore$ sub (7) in (6) we get

$$\frac{\sigma}{\varepsilon}\rho_v = -\frac{\partial \rho_v}{\partial t} \Rightarrow \frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0 \quad \text{---} (8)$$

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0 \quad \text{--- (8)}$$

The above equation is of the form

$$\frac{\partial x}{\partial t} + a x = 0$$

solution of this equation is $x = x_0 e^{-at}$

where $x_0 \rightarrow$ Initial condition

$|||^{ly}$ solution of Equation (8) is

$$\rho_v = \rho_0 e^{-\sigma/\varepsilon \, t}.$$

$$\boxed{\rho_v = \rho_0 e^{-t/\tau}}$$ where ρ_0 = charge density at $t = 0$.

--- (9)

This shows that if there is a temporary imbalance of electrons inside the given material, the charge density decays exponentially with time constant $\tau = \varepsilon/\sigma$ sec. This time is called relaxation time.

The relaxation time (τ) is defined as the time required by the charge density to decay to 36.8% of its initial value

$$\tau = \text{Relaxation time} = \frac{\varepsilon}{\sigma} \text{ sec.}$$

DIELECTRIC MATERIALS:-

→ It is seen that the conductors have large number of free electrons while insulators and dielectric materials do not have free charges.

→ The charges in dielectrics are bound by the finite forces and hence called bound charges. As they are bound & not free, they cannot contribute to the conduction Process.

→ But if subjected to an Electric field $\vec{E}$, they shift their relative positions, against the normal molecular and atomic forces. This shift in the relative positions of bound charges, allows the dielectric to store the energy.

The shifts in positive and negative charges are in opposite directions and under the influence of an applied electric field $\vec{E}$ such charges act like small electric dipoles.

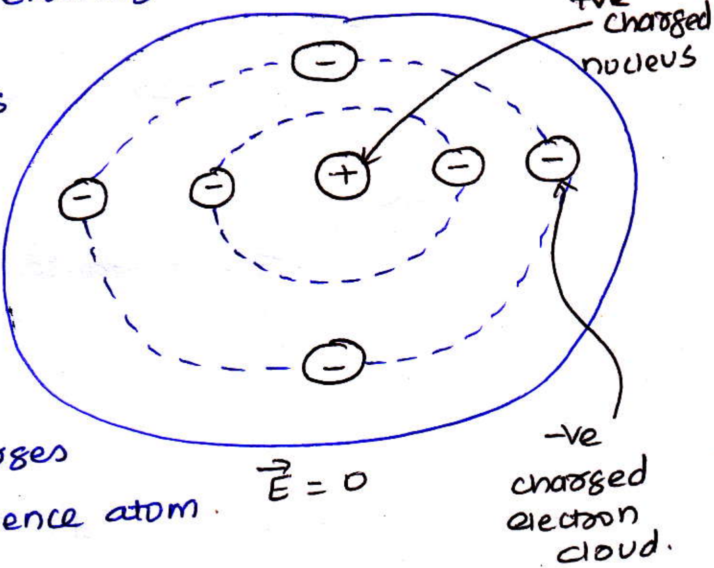
→ These electric dipoles produce an electric field which opposes the externally applied Electric field. This process, due to which separation of bound charges results to produce electric dipoles, under the influence of electric field $\vec{E_r}$ is called POLARIZATION.

POLARIZATION :

→ consider an atom of a dielectric

→ This consists of a nucleus with +ve charge and -ve charge in the form of revolving electrons in the orbits.

→ The negative charge is thus considered to be in the form of cloud of electrons. as shown in figure.

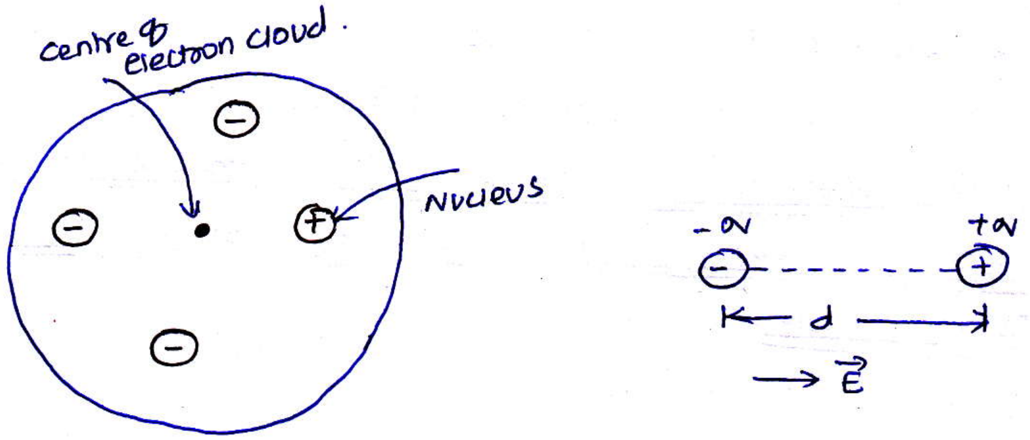


Note that $\vec{E}$ applied is zero. The number of +ve charges is same as -ve charges and hence atom is electrically neutral.

Due to symmetry, both +ve and -ve charges can be assumed to be point charges q equal amount, coinciding at the centre. Hence there cannot exists an electric dipole. This is called unpolarized atom.

When electric field $\vec{E}$ is applied, the symmetrical distribution of charges gets disturbed. The +ve charge experience a force $\vec{F} = Q\vec{E}$ while the -ve charge experience a force $\vec{F} = -Q\vec{E}$ in the opposite direction.

Now there is seperation b/w the nucleus and the centre of the electron cloud as shown in fig below. such an atom is called POLARIZED ATOM:



MATHEMATICAL EXPRESSION FOR POLARIZATION:

When the dipole is formed due to Polarization, there exists an electric dipole moment $\vec{P}$

$$\vec{P} = Q\vec{d} \quad ---(1)$$

where Q = Magnitude of one of the two charges

$\vec{d}$ = Distance vector from -ve to +ve charge

Let

$n \rightarrow$ No. of dipoles per unit volume

$\Delta V \rightarrow$ Total volume of the dielectric

$N \rightarrow$ Total dipole $= n\Delta V$

Then the total dipole moment is to be obtained by using superposition theorem.

$$\vec{P}_{total} = Q_1\vec{d_1} + Q_2\vec{d_2} + \dots + Q_n\vec{d_n}$$

$$\vec{P}_{total} = \sum_{i=1}^{n\Delta V} Q_i\vec{d_i} \quad ---(2)$$

~~As~~ The Polarization $\vec{P}$ is defined as the total dipole moment per unit volume.

$$\boxed{\therefore \vec{P} = \lim_{\Delta V \to 0} \frac{\sum_{i=1}^{n\Delta V} Q_i\vec{d_i}}{\Delta V}} \quad ---(3)$$

It is measured in (C/m^2)

It can be seen that the units of Polarization are same as that of flux density $\vec{D}$. Thus Polarization increases the electric flux density in a dielectric medium. Hence we can write, flux density in an dielectric as

$$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$$

DIELECTRIC STRENGTH:

→ The ideal dielectric is non conducting but practically no dielectric can be ideal.

→ As the electric field applied to dielectric increases sufficiently, due to the force exerted on to the molecules, the electrons in the dielectric become free.

→ Under such large Electric field, the dielectric becomes conducting due to presence of large number of free electrons. This condition of dielectric is called dielectric breakdown.

The minimum value of the applied electric field at which the dielectric breaksdown is called dielectric strength of that dielectric.

It is measured in V/m or kV/cm.

BOUNDARY CONDITIONS:

→ When an electric field passes from one medium to another medium, it is important to study the conditions at the boundary b/w the two media.

→ The conditions existing at the boundary of the two media, when field passes from one medium to other are called boundary conditions.

Depending upon the nature of the media, there are two situations of the boundary conditions.

1. Boundary b/w conductor and free space
2. Boundary b/w two dielectrics with different properties.

→ The free space is nothing but a dielectric, hence first case is nothing but the boundary b/w conductors and dielectric.

→ For studying the boundary conditions, the maxwell's equation for Electrostatics are required.

$$\boxed{\oint \vec{E} \cdot \vec{dL} = 0 \quad \& \quad \oint \vec{D} \cdot \vec{ds} = Q}$$

similarly the field intensity $\vec{E}$ is required to be decomposed into two components namely

(i) Tangential to the boundary $[\vec{E_{tan}}]$ &
(ii) Normal to the boundary $[\vec{E_N}]$

$$\boxed{\therefore \quad \vec{E} = \vec{E}_{tan} + \vec{E_N}}$$

III^r decomposition is required for flux density $\vec{D}$ as well.

BOUNDARY CONDITIONS B/W CONDUCTOR & FREE SPACE

Consider a boundary between conductor and free space. The conductor is ideal having infinite conductivity. For ideal conductors it is known that.

1. The field intensity inside a conductor is zero and the flux density inside a conductor is zero.

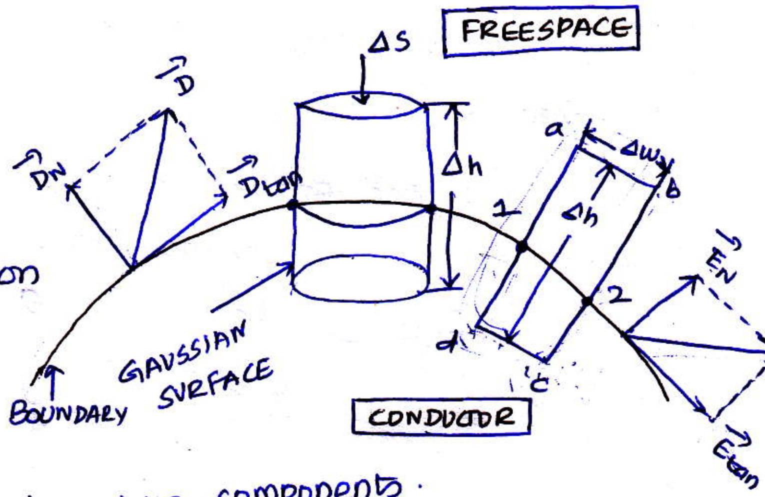
2. No charge can exist within a conductor. The charge appears on the surface in the form of surface charge density.

3. The charge density with in the conductor is zero.

thus $\vec{E}$, $\vec{D}$ and ρ_v within the conductor is zero. while ρ_s → surface charge density on the surface of the conductor.

To determine the boundary conditions let us use the closed path and the Gaussian surface. (9)

Consider the conductor free space boundary as shown in fig below.



$\vec{E}$ at BOUNDARY:

Let $\vec{E}$ be the Electric field intensity, in the direction shown in the figure, making some angle with the boundary.

This $\vec{E}$ can be resolved into two components.

1. The component tangential to the surface ($\vec{E}_{tan}$)
2. The component normal to the surface [$\vec{E}_N$]

It is known that

$$\oint \vec{E} \cdot \vec{dL} = 0$$

The integral of $\vec{E} \cdot \vec{dL}$ carried over a closed contour is zero. i.e work done in carrying a unit +ve charge along a closed path is zero.

Consider a rectangular closed path abcda as shown in fig. It is traced in clockwise direction as a-b-c-d-a and hence $\oint \vec{E} \cdot \vec{dL}$ can be divided into four parts.

$$\oint \vec{E} \cdot \vec{dL} = \int_a^b \vec{E} \cdot \vec{dL} + \int_b^c \vec{E} \cdot \vec{dL} + \int_c^d \vec{E} \cdot \vec{dL} + \int_d^a \vec{E} \cdot \vec{dL} = 0.$$

The closed contour is placed in such a way that its two sides a-b and c-d are $\parallel^{el}$ to tangential direction to the surface while the other two are normal to the surface, at the boundary.

The rectangle is an elementary rectangle with elementary height Δh and elementary width ΔW. The rectangle is place in such a way that half of its is in the conductor and remaining half is in the free space.

Thus $\Delta h/2$ is in the conductor and $\Delta h/2$ is in the free space.

Now the portion c-d is in the conductor where $\vec{E}=0$ hence the corresponding integral is zero

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} + \int_b^c \vec{E}\cdot\vec{dL} + \int_d^a \vec{E}\cdot\vec{dL} = 0 \quad ---- \text{(A)}$$

As the width Δw is very small, $\vec{E}$ over it can be assumed constant and hence can be taken out for integration.

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} = \vec{E}\int_a^b \vec{dL} = \vec{E}\,(\Delta W)$$

But ΔW is along tangential direction to the boundary in which direction $\vec{E} = \vec{E_{tan}}$

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} = E_{tan}\,(\Delta W) \quad \text{where } E_{tan} = |E_{tan}| \quad --- \text{(1)}$$

Now b-c is $\|^{d}$ to the normal component so we have $\vec{E}=\vec{E_N}$ along this direction, Let $E_N = |\vec{E_N}|$

over the small height Δh, E_N can be assumed Constant and can be taken out of integration.

$$\therefore \int_b^c \vec{E}\cdot\vec{dL} = \vec{E}\int_b^c \vec{dL} = \vec{E_N}\int_b^c \vec{dL}$$

But out of b-c, b-2 is in free space and 2-c is in the conductor where $\vec{E}=0$

$$\therefore \int_b^c \vec{dL} = \int_b^2 \vec{dL} + \int_2^c \vec{dL} = \frac{\Delta h}{2} + 0 = \frac{\Delta h}{2}$$

$$\therefore \int_b^c \vec{E}\cdot\vec{dL} = E_N\left(\frac{\Delta h}{2}\right) \text{ --- } (2)$$

similarly for path d-a, the condition is same as for the path b-c, only direction is opposite

$$\therefore \int_d^a \vec{E}\cdot\vec{dL} = -E_N\left(\frac{\Delta h}{2}\right) \text{ --- } (3)$$

sub (1), (2) and (3) in (A) we get

$$E_{tan}(\Delta W) + E_N\left(\frac{\Delta h}{2}\right) - E_N\left(\frac{\Delta h}{2}\right) = 0$$

$$E_{tan}(\Delta W) = 0 \qquad \because \Delta W \neq 0 \text{ as finite}$$

$$\Rightarrow \boxed{E_{tan} = 0}$$

Thus the tangential component of the electric field intensity is zero at the boundary b/w conductor and free space.

D_N at the BOUNDARY

To find normal component of $\vec{D}$, select a closed Gaussian surface in the form of right circular cylinder as shown in the figure.

It's height is Δh and is placed in such a way that $\Delta h/2$ is in the conductor and remaining $\Delta h/2$ is in the free space. It's axis in the normal direction to the surface.

According to Gauss's law

$$\oint_S \vec{D}\cdot\vec{ds} = Q.$$

The surface integral must be evaluated over three surfaces (i) TOP (ii) Bottom (iii) Lateral.

Let the area of top and bottom is same equal to Δs

$$\therefore \int_{top} \vec{D}\cdot\vec{ds} + \int_{bottom} \vec{D}\cdot\vec{ds} + \int_{lateral} \vec{D}\cdot\vec{ds} = Q \quad \text{---} \ (1)$$

The bottom surface is in the conductor where $\vec{D} = 0$ hence corresponding integral is zero

The top surface is in the free space and we are interested in the boundary condition, hence top surface can be shifted at the boundary with $\Delta h \to 0$.

$$\therefore \int_{top} \vec{D}\cdot\vec{ds} + \int_{lateral} \vec{D}\cdot\vec{ds} = Q \quad \text{---} \ (2)$$

The lateral surface area is $2\pi r \Delta h$

where $r \to$ radius of the cylinder

But $\Delta h \to 0$, this area reduces to zero and corresponding integral is zero.

While only component of $\vec{D}$ present is the normal component having magnitude D_N. The top surface is very small over which D_N can be assumed constant and can be taken out of integration.

$$\therefore \int_{top} \vec{D}\cdot\vec{ds} = D_N \int_{top} \vec{ds} = D_N \Delta s \quad \text{---} \ (3)$$

$$\therefore D_N \Delta s = Q \quad \text{---} \ (4)$$

But at Boundary, ~~condition~~ the charge exists in the form of surface charge density $\rho_s \ C/m^2$

$$\therefore Q = \rho_s \Delta s \quad \text{----} \ (5)$$

sub 5 in 4 we get

$$D_N \Delta S = \rho_s \Delta S$$

$$\therefore \boxed{D_N = \rho_s}$$

Thus the flux leaving normally and the normal component of flux density is equal to the surface charge density.

$$\therefore D_N = \varepsilon_0 E_N = \rho_s$$

$$\therefore E_N = \frac{\rho_s}{\varepsilon_0}.$$

BOUNDARY CONDITION B/W CONDUCTOR & DIELECTRIC

The free space is a dielectric with $\varepsilon = \varepsilon_0$. Thus if the boundary is between conductor and dielectric $\varepsilon = \varepsilon_0 \varepsilon_r$.

$$\therefore \boxed{\begin{array}{l} E_{tan} = D_{tan} = 0 \\ D_N = \rho_s \\ E_N = \dfrac{\rho_s}{\varepsilon} = \dfrac{\rho_s}{\varepsilon_0 \varepsilon_r}. \end{array}}$$

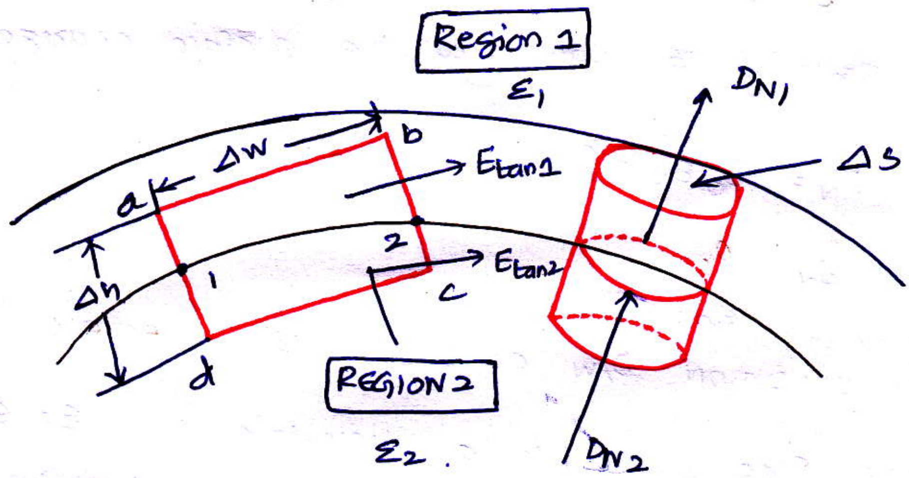
BOUNDARY CONDITIONS BETWEEN TWO PERFECT DIELECTRICS:

Let us consider the boundary b/w two perfect dielectrics. one dielectric has permittivity ε_1 while other has permittivity ε_2. The interface is shown in the figure.

The $\vec{E}$ and $\vec{D}$ are to be obtained again by resolving each into two components, tangential to the boundary and normal to the surface.

Consider a closed loop abcda rectangular in shape having elementary heigh Δh and elementary width Δw, as shown in figure.

It is placed in such a way that $\Delta h/2$ is in the dielectric 1 while the remaining is dielectric 2. Let us evaluate the integral $\vec{E} \cdot \vec{dL}$ along this path, tracing it in clockwise direction as a-b-c-d-a.



$$\oint \vec{E} \cdot \vec{dL} = 0 \quad \text{---} (1)$$

$$\therefore \int_a^b \vec{E} \cdot \vec{dL} + \int_b^c \vec{E} \cdot \vec{dL} + \int_c^d \vec{E} \cdot \vec{dL} + \int_d^a \vec{E} \cdot \vec{dL} = 0 \quad \text{---} (2)$$

Now $\vec{E_1} = \vec{E_{1t}} + \vec{E_{1N}}$

$\vec{E_2} = \vec{E_{2t}} + \vec{E_{2N}}$

Both $\vec{E_1}$ and $\vec{E_2}$ in the respective dielectrics have both the components, normal and tangential.

Let $|\vec{E_{1t}}| = E_{tan1}$ $\quad |\vec{E_{2t}}| = E_{tan2}$

$|\vec{E_{1N}}| = E_{1N}$ $\quad |\vec{E_{2N}}| = E_{2N}$.

Now for the rectangle to be reduced at the surface to analyse boundary conditions, $\Delta h \to 0$

As $\Delta h \to 0$ $\int_b^c$ and $\int_d^a$ become zero as these are line integrals along Δh and $\Delta h \to 0$, Hence eqn (2) becomes

$$\int_a^b \vec{E}\cdot\vec{dL} + \int_c^d \vec{E}\cdot\vec{dL} = 0 \quad \text{--- (3)}$$

Now a-b is in dielectric 1 hence the corresponding component of $\vec{E}$ is E_{tan1} as a-b direction is tangential to the surface

$$\therefore \int_a^b \vec{E}\cdot\vec{dL} = E_{tan1}\int_a^b \vec{dL} = E_{tan1}(\Delta W) \quad \text{--- (4)}$$

while c-d is in dielectric 2 hence the corresponding component of $\vec{E}$ is E_{tan2} as c-d direction is also tangential to the surface. But the direction of c-d is opposite to a-b hence corresponding integral is negative of the integral obtained for path a-b.

$$\therefore \int_c^d \vec{E}\cdot\vec{dL} = -E_{tan2}(\Delta W) \quad \text{---- (5)}$$

substituting (4) and (5) in (3) we get

$$E_{tan1}(\Delta W) - E_{tan2}(\Delta W) = 0$$

$$\Rightarrow \boxed{E_{tan1} = E_{tan2}} \quad \text{--- (6)}$$

Thus the tangential component of field intensity at the boundary in both the dielectrics remain same i.e Electric field intensity is continuous across the boundary

The relation b/w $\vec{D}$ and $\vec{E}$ is known as,

$$\vec{D} = \varepsilon\vec{E}$$

Hence if D_{tan1} and D_{tan2} are magnitudes of the tangential components of $\vec{D}$ in dielectric 1 and 2 respectively then,

$$D_{tan1} = \varepsilon_1 E_{tan1}$$

$$D_{tan2} = \varepsilon_2 E_{tan2}$$

$$\frac{D_{tan1}}{\varepsilon_1} = \frac{D_{tan2}}{\varepsilon_2}$$

$$\boxed{\frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}} \quad \text{---- } (7)$$

Thus tangential components of $\vec{D}$ undergoes some change across the interface hence tangential $\vec{D}$ is said to be discontinuous across the boundary.

To find the normal components, let us use Gauss's law. Consider a Gaussian surface in the form of right circular cylinder, placed in such a way that half of it lies in dielectric 1 while the remaining half in dielectric 2. The height $\Delta h \to 0$ hence flux leaving from its lateral surface is zero. The surface area of its top and bottom is ΔS.

$$\therefore \oint \vec{D} \cdot \vec{dS} = Q \quad \text{--- } (8)$$

$$\therefore \left[\int_{top} + \int_{bottom} + \int_{lateral} \right] \vec{D} \cdot \vec{dS} = Q \quad \text{---- } (9)$$

But $$\int_{lateral} \vec{D} \cdot \vec{dS} = 0 \quad \text{as } \Delta h \to 0 \text{ ---- } (10)$$

$$\therefore \int_{top} \vec{D} \cdot \vec{dS} + \int_{bottom} \vec{D} \cdot \vec{dS} = Q \text{. ---- } (11)$$

The flux leaving normal to the boundary is normal to the top and bottom surfaces.

$$\therefore |\vec{D}| = D_{N1} \text{ for dielectric } 1.$$

$$= D_{N2} \text{ for dielectric } 2.$$

As the top and bottom surfaces are elementary, flux density can be assumed constant and can be taken out of integration

$$\therefore \int_{top} \vec{D}\cdot\vec{ds} = D_{N1} \int_{top} \vec{ds} = D_{N1}\,\Delta S \text{ ---- } (12)$$

For top surface, the direction of D_N is entering the boundary while for bottom surface, the direction of D_N is leaving the boundary.

Both are opposite in direction, at the boundary

$$\therefore \int_{bottom} \vec{D}\cdot\vec{ds} = -D_{N2} \int_{bottom} \vec{ds} = -D_{N2}\,\Delta S \text{ --- } (13)$$

Sub (12) and (13) in (11) we get

$$D_{N1}\,\Delta S - D_{N2}\,\Delta S = Q$$

But $Q = \rho_s \Delta S$

$$\boxed{D_{N1} = D_{N2}}$$

$$\therefore \boxed{D_{N1} - D_{N2} = \rho_s}$$

There is no free charge available in perfect dielectric and hence no free charge can exist on the surface. All charges in dielectric are bound charges and are not free.

Hence at ideal dielectric media boundary the surface charge density ρ_s can be assumed zero.

$\therefore \rho_s = 0$

$D_{N1} - D_{N2} = 0$

$$\boxed{D_{N1} = D_{N2}}$$

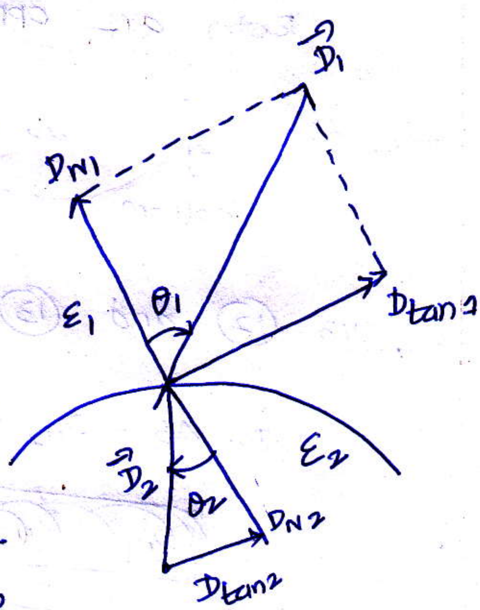
Hence the normal component of flux density $\vec{D}$ is continuous at the boundary b/w the two perfect dielectrics.

$\therefore D_{N1} = \varepsilon_1 E_{N1}$ and $D_{N2} = \varepsilon_2 E_{N2}$

$\therefore \frac{D_{N1}}{D_{N2}} = \frac{\varepsilon_1}{\varepsilon_2} \frac{E_{N1}}{E_{N2}} = 1.$

$$\boxed{\therefore \frac{E_{N1}}{E_{N2}} = \frac{\varepsilon_2}{\varepsilon_1} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}}$$

Refraction of $\vec{D}$ at the Boundary:



The directions of $\vec{D}$ and $\vec{E}$ change at the boundary b/w the two dielectrics.

Let $\vec{D_1}$ and $\vec{E_1}$ make an angle θ_1 with the normal to the surface.

$\vec{D_1}$ and $\vec{E_1}$ direction is same as

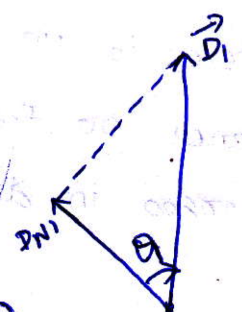
$$\boxed{\vec{D_1} = \varepsilon_1 \vec{E_1}}$$

This is shown in the figure.

Let $|\vec{D_1}| = D_1$

$|\vec{D_2}| = D_2$

from fig



$\cos\theta_1 = \frac{D_{N1}}{D_1}$

$\therefore D_{N1} = D_1 \cos\theta_1$ --- (1)

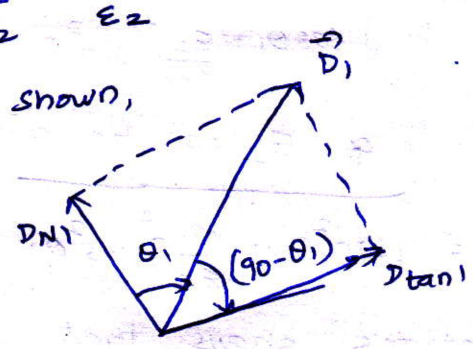
$|||^{ly}$ $D_{N2} = D_2 \cos\theta_2$ --- (2)

But W.K.T $D_{N1} = D_{N2}$

$\therefore D_1 \cos\theta_1 = D_2 \cos\theta_2$

$|||^{ly}$ W.K.T $\dfrac{D_{tan1}}{D_{tan2}} = \dfrac{\varepsilon_1}{\varepsilon_2}$

From the Figure shown,



$\cos(90-\theta_1) = \dfrac{D_{tan1}}{D_1}$

$\sin\theta_1 = \dfrac{D_{tan1}}{D_1}$

$$\left.\begin{aligned} \Rightarrow D_{tan1} &= D_1 \sin\theta_1 \\ |||^{ly} \quad D_{tan2} &= D_2 \sin\theta_2 \end{aligned}\right\} \text{--- (3)}$$

$$\frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{D_{tan1}}{D_{tan2}} \quad \text{--- (A)}$$

Now (3a) $\div$ (1)

$$\frac{D_1 \sin\theta_1}{D_1 \cos\theta_1} = \frac{D_{tan1}}{D_{N1}} \Rightarrow \tan\theta_1 = \frac{D_{tan1}}{D_{N1}} \quad \text{--- (4a)}$$

$|||^{ly}$ $\tan\theta_2 = \dfrac{D_{tan2}}{D_{N2}}$ --- (4b)

$$\frac{(4a)}{(4b)} \Rightarrow \frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}} \, \frac{D_{N2}}{D_{N1}}$$

But $D_{N1} = D_{N2}$

$$\therefore \frac{\tan\theta_1}{\tan\theta_2} = \frac{D_{tan1}}{D_{tan2}} = \frac{\varepsilon_1}{\varepsilon_2}$$

$$\Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$$

This is called law of refraction. Thus the angles θ_1 and θ_2 are dependent on permittivities of two media and not on $\vec{D}$ or $\vec{E}$.

If $\varepsilon_1 > \varepsilon_2$ then $\theta_1 > \theta_2$

The magnitude of $\vec{D}$ in region 2 can be obtained as

$$D_2^2 = D_{N2}^2 + D_{tan2}^2 = (D_1 \cos\theta_1)^2 + D_{tan2}^2$$

Now $D_{tan2} = D_2 \sin\theta_2 = \frac{D_1 \sin\theta_1 \varepsilon_2}{\varepsilon_1}$. [from (A)]

$$\therefore \quad D_2^2 = (D_1 \cos\theta_1)^2 + \left(D_1 \sin\theta_1 \frac{\varepsilon_2}{\varepsilon_1}\right)^2$$

$$D_2 = D_1 \sqrt{\cos^2\theta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \sin^2\theta_1}.$$

III^ly Magnitude of E_2 can be obtained as

$$E_2 = E_1 \sqrt{\sin^2\theta_1 + \left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \cos^2\theta_1}$$

the equations shows that

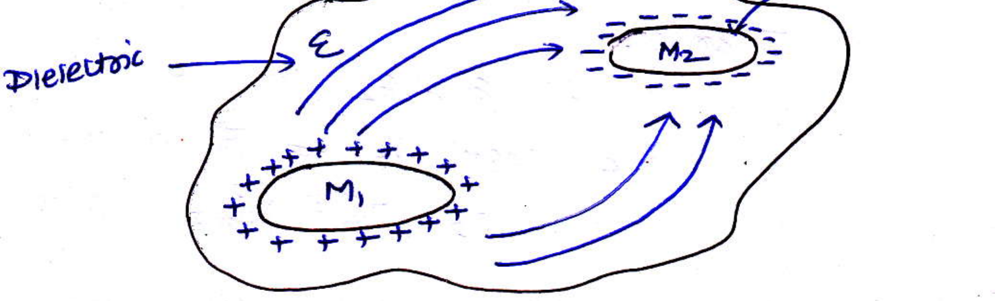
i) $\vec{D}$ is larger in region of larger Permittivity
ii) $\vec{E}$ is larger in region of smaller permittivity
iii) $|\vec{D_1}| = |\vec{D_2}| \Rightarrow$ if $\theta_1 = \theta_2 = 0°$
(iv) $|\vec{E_1}| = |\vec{E_2}|$ if $\theta_1 = \theta_2 = 90°$.

To find the angles θ_1 and θ_2, w.r.to normal use the dot Product if normal direction to the boundary is known.

CONCEPT OF CAPACITANCE:-

Consider two conducting materials M_1 and M_2 which are placed in a dielectric medium having permittivity ε. The material M_1 carries a Positive charge Q while the material M_2 carries a Negative charge $-Q$ (equal in magnitude). There are no other charges present and the total charge of the system is zero.

In conductors, charge cannot reside within the conductor and it resides only on the surface. Thus for M_1 and M_2 charges $+Q$ and $-Q$ resides on the surfaces of M_1 and M_2 respectively. This is shown in fig below.



Such a system which has two conducting surfaces carrying equal and opposite charges seperated by a dielectric is called <u>capacitive systems</u> giving rise to a <u>capacitance</u>.

The electric field is normal to the conductor surface and the ~~electric~~ flux is directed from M_1 towards M_2 in such a system. There exists a potential difference b/w the two surfaces M_1 and M_2. Let this potential is V_{12}.

The ratio of magnitudes of the total charge on any one of the two conductors and potential difference b/w the conductors is called the capacitance. It is denoted by 'c'.

$$c = \frac{Q}{V_{12}}$$

$V_{12} = V_2 - V_1$
$V_2 = -ve$
$V_1 = +ve$

In general $c = \frac{Q}{V}$

where Q = charge in coulombs
V = Potential difference in volts.

The capacitance is measured in farads (F) and

$$1 \text{ Farad} = \frac{1 \text{ coulombs}}{1 \text{ volt}}.$$

As charge Q resides only on the surface of the conductor it can be obtained ~~from~~ the Gauss's law as,

$$Q = \oint_S \vec{D}\cdot\vec{ds} = \oint_S \varepsilon_0\varepsilon_r\vec{E}\cdot\vec{ds} = \oint_S \varepsilon\vec{E}\cdot\vec{ds}$$

while V is the work done in moving unit positive charge from -ve to +ve surface and can be obtained as,

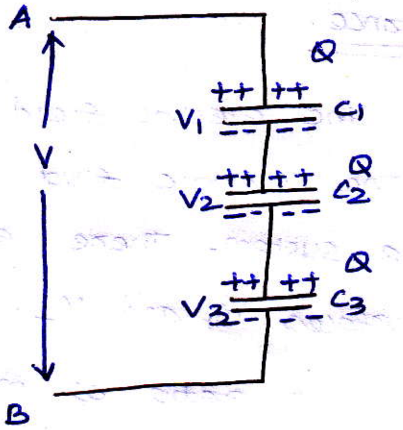
$$V = -\int_L \vec{E}\cdot\vec{dL} = -\int_-^+ \vec{E}\cdot\vec{dL}$$

Hence capacitance can be expressed as

$$c = \frac{Q}{V} = \frac{\oint_S \varepsilon\vec{E}\cdot\vec{ds}}{-\int_-^+ \vec{E}\cdot\vec{dL}} \text{ F}.$$

CAPACITORS IN SERIES:

consider the three capacitors in series connected across the applied voltage V as shown in figure below.

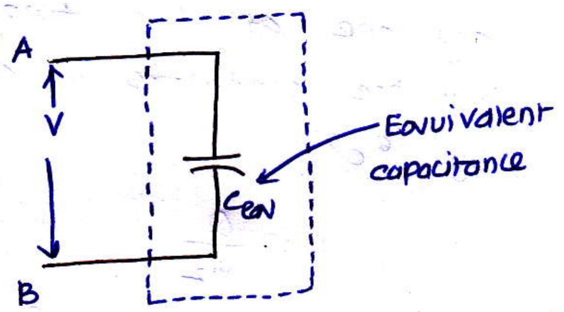


$$Q = C_1V_1 = C_2V_2 = C_3V_3.$$

Giving

$$V_1 = \frac{Q}{C_1}\ ;\ V_2 = \frac{Q}{C_2}\ ;\ V_3 = \frac{Q}{C_3}$$

If an equivalent capacitor also stores the same charge, when applied with the same voltage, then it is obvious that,



$$C_{eq} = \frac{Q}{V} \text{ (or) } \cancel{\frac{C_{eq}}{Q}}$$

$$V = \frac{Q}{C_{eq}}$$

But $V = V_1 + V_2 + V_3$

$$\frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$$\therefore \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

It is easy to find V_1, V_2 and V_3 if Q is known

For 'n' capacitors in series

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}}$$

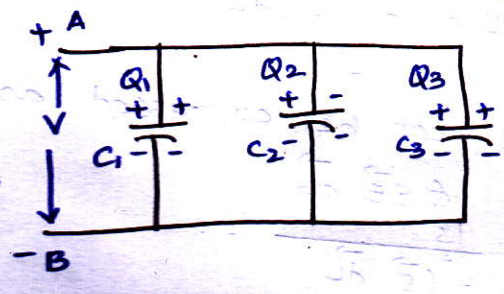
CAPACITORS IN PARALLEL:

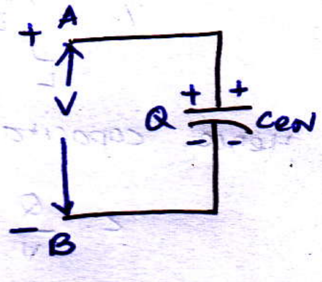
When capacitors are in $\parallel^{el}$, the same voltage exists across them, but charges are different.

$Q_1 = C_1V$

$Q_2 = C_2V$

$Q_3 = C_3V$





The Total charge stored by the Parallel bank of capacitors Q is given by

$$Q = Q_1 + Q_2 + Q_3$$
$$= C_1 V + C_2 V + C_3 V$$
$$C_{eq} V = (C_1 + C_2 + C_3) V$$
$$\therefore \boxed{C_{eq} = C_1 + C_2 + C_3}$$

For n capacitors in $\parallel^{el}$

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \dots + C_n}$$

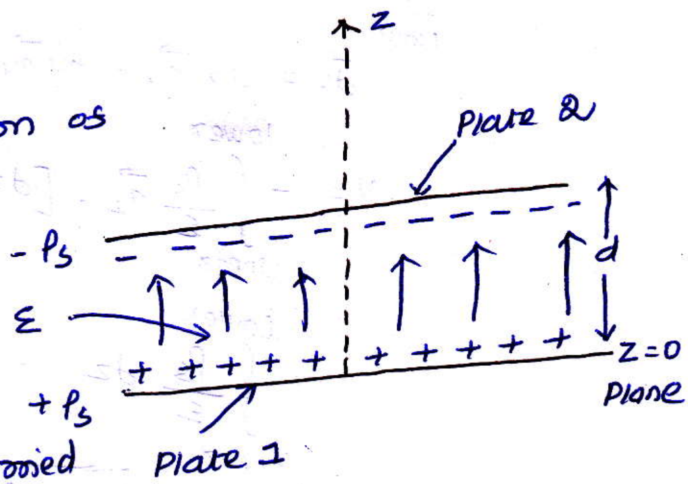
PARALLEL PLATE CAPACITORS:-

A Parallel Plate capacitor is shown in figure. It consists of two parallel metallic plates seperated by a distance 'd'. The space b/w the plates is filled with a dielectric of Permittivity ε.

The lower plate, Plate 1 carries the positive charge and is distributed over it with a charge density $+\rho_s$. The upper Plate, Plate 2 carries the negative charge and is distributed over its surface with a charge density of $-\rho_s$. The plate 1 is Placed in $z=0$ i.e xy plane hence normal to it is z-direction. while upper plate 2 is in $z=d$ Plane, Parallel to xy plane.

Let A = Area of cross section of the Plates in m^2.

$$\boxed{\therefore Q = \rho_s A} \text{ coulombs.}$$



This is magnitude of charge carried on any one plate as charge carried by both is equal in magnitude.

To find Potential difference, let us obtain $\vec{E}$ b/w the Plates.

Assuming Plate 1 to be infinite sheet of charge

$$\vec{E_1} = \frac{\rho_s}{2\varepsilon}\vec{a_N} \quad \text{[Page No 16 of unit-II]}.$$

$$= \frac{\rho_s}{2\varepsilon}\vec{a_z} \text{ V/m}.$$

The $\vec{E_1}$ is normal at the boundary b/w conductor and dielectric without any tangential component.

While for plate 2, we can write

$$\vec{E_2} = \frac{-\rho_s}{2\varepsilon}\vec{a_N} = \frac{-\rho_s}{2\varepsilon}(-\vec{a_z}) = \text{V/m}.$$

The direction of $\vec{E_2}$ is downwards i.e in $-\vec{a_z}$ direction.

∴ In b/w Plates

$$\vec{E} = \vec{E_1} + \vec{E_2}$$

$$= \frac{\rho_s}{2\varepsilon}\vec{a_z} + \frac{\rho_s}{2\varepsilon}\vec{a_z} \Rightarrow \frac{\rho_s}{\varepsilon}\vec{a_z}$$

$$\boxed{\vec{E} = \frac{\rho_s}{\varepsilon}\vec{a_z}}$$

The Potential difference is given by.

$$V = -\int_{-}^{+} \vec{E}\cdot\vec{dL} = -\int_{\text{upper}}^{\text{lower}} \frac{\rho_s}{\varepsilon}\vec{a_z}\cdot\vec{dL}$$

Now $\vec{dL} = dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}$ in cartesian system,

$$V = -\int_{\text{upper}}^{\text{lower}} \frac{\rho_s}{\varepsilon}\vec{a_z}\cdot[dx\,\vec{a_x} + dy\,\vec{a_y} + dz\,\vec{a_z}]$$

$$= -\int_{\text{upper}}^{\text{lower}} \frac{\rho_s}{\varepsilon}\,dz = -\frac{\rho_s}{\varepsilon}[z]_d^0 = -\frac{\rho_s}{\varepsilon}(-d) = \frac{\rho_s d}{\varepsilon}.$$

$$\boxed{V = \frac{\rho_s d}{\varepsilon}} \text{ Volts}$$

$\therefore$ The capacitance is the ratio of charge Q to Voltage V.

$$C = \frac{Q}{V} = \frac{\rho_s A}{\frac{\rho_s d}{\varepsilon}} = \frac{\varepsilon A}{d} \cdot F$$

Thus is $\varepsilon = \varepsilon_0 \varepsilon_r$

$$\boxed{C = \frac{\varepsilon_0 \varepsilon_r A}{d}} \ F$$

It can be seen that the value of capacitance depends on

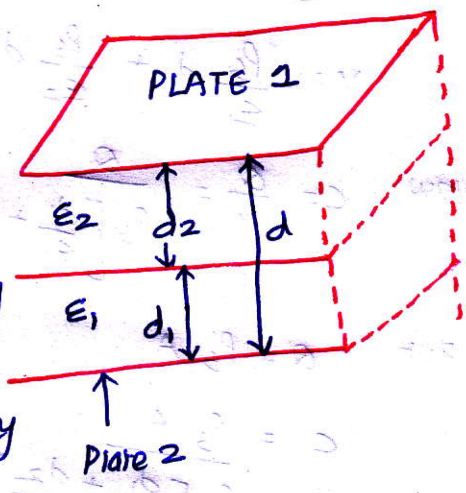
(i) The permittivity of the dielectric used
(ii) The area of cross section of the plates
(iii) The distance of seperation of plates.

COMPOSITE PARALLEL PLATE CAPACITORS:

The composite parallel plate capacitor is one in which the space b/w the plates is filled with more than one dielectric.

Consider a composite capacitor with space filled with two seperate dielectrics for distance d_1 and d_2

The dielectric interface is parallel to the conducting Plates.



The space d_1 is filled with dielectric having Permittivity ε_1 while space d_2 is filled with dielectric having Permittivity ε_2

Let Q = charge on each plate

$\vec{E_1}$ = Field intensity in region d_1

$\vec{E_2}$ = Field intensity in region d_2

Both the intensities are uniform

$\therefore V_1 = E_1 d_1$

$V_2 = E_2 d_2$

where E_1 and E_2 are the magnitudes of the two intensities.

$$V = V_1 + V_2 = E_1 d_1 + E_2 d_2 \quad \text{--- (1)}$$

At a dielectric - dielectric interface, the normal components of flux densities are equal

i.e $D_{N1} = D_{N2}$

Now $D_1 = \varepsilon_1 E_1$ and $D_2 = \varepsilon_2 E_2$

$$\Rightarrow E_1 = \frac{D_1}{\varepsilon_1} \;\&\; E_2 = \frac{D_2}{\varepsilon_2} \quad \text{--- (2)}$$

sub (2) in (1) we get

$$V = \frac{D_1}{\varepsilon_1} d_1 + \frac{D_2}{\varepsilon_2} d_2 \quad \text{--- (3)}$$

The magnitude of surface charge is same on each plate hence

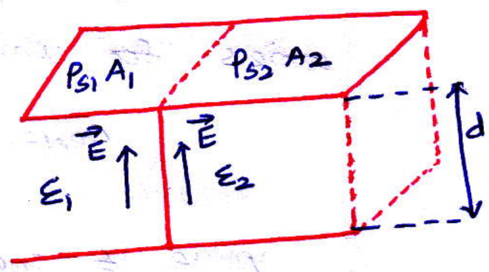
$$\rho_s = D_1 = D_2 \quad \text{--- (4)}$$

sub (4) in (3) we get

$$V = \frac{\rho_s}{\varepsilon_1} d_1 + \frac{\rho_s}{\varepsilon_2} d_2 \Rightarrow \rho_s \left[\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2} \right] \quad \text{--- (5)}$$

Q --- (6)

DIELECTRIC BOUNDARY NORMAL TO THE PLATES:



Consider the composite capacitor in which dielectric boundary is normal to the conducting plates.

The dielectric ε_1 occupying area A_1 of the plates, while dielectric ε_2 occupying area A_2 as shown in the figure.

The total potential across the two plates is V and the distance b/w the plates is d. Hence magnitude of $\vec{E}$ is

$$E = \frac{V}{d}$$

$\therefore$ At the boundary, both $\vec{E}_1$ and $\vec{E}_2$ are tangential and for dielectric - dielectric interface tangential components are equal.

$$E_{tan1} = E_{tan2} = E_1 = E_2 = \frac{V}{d} \quad \text{---} (1)$$

Now $D_1 = \varepsilon_1 E_1$ & $D_2 = \varepsilon_2 E_2$ --- (2)

sub (1) in (2),

$$D_1 = \frac{\varepsilon_1 V}{d} \quad \text{------} \quad D_2 = \frac{\varepsilon_2 V}{d} \quad \text{---} (3)$$

On the plates the charge is divided into two parts

on area A_1, the charge density is $\rho_{S1} = D_1$ while
on area A_2, the charge density is $\rho_{S2} = D_2$ } --- (4)

$\therefore$ $Q = Q_1 + Q_2$ --- (5)

$= \rho_{S1} A_1 + \rho_{S2} A_2$ --- (6)

sub (4) in (6)

$= D_1 A_1 + D_2 A_2$ -- (7)

sub (3) in (7)

$$= \frac{\varepsilon_1 V A_1}{d} + \frac{\varepsilon_2 V A_2}{d}$$

$$C = \frac{Q}{V} = \frac{\frac{\varepsilon_1 V A_1 + \varepsilon_2 V A_2}{d}}{V} \Rightarrow \frac{\varepsilon_1 A}{d} + \frac{\varepsilon_2 A}{d}$$

$$\boxed{C = C_1 + C_2}$$

where $C_1 = \frac{\varepsilon_1 A}{d}$ $C_2 = \frac{\varepsilon_2 A}{d}$.

Thus is dielectric boundary is $\parallel^{el}$ to the plates, the arrangement is equivalent to two capacitors in series for which

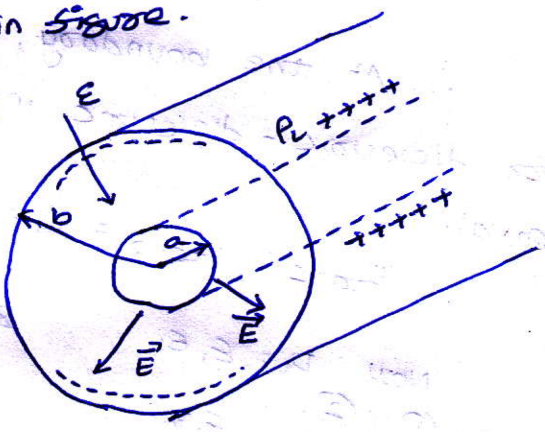
$$C_{eq} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

while is the dielectric boundary is normal to the plates, the arrangement is equivalent to two capacitors in $\parallel^{el}$ for which

$$C_{eq} = C_1 + C_2 .$$

CAPACITANCE OF A CO-AXIAL CABLE:-

Consider a co-axial cable or co-axial capacitor as shown in figure.



The two concentric conductors are seperated by dielectric of Permittivity ε.

The length of the cable is L meters.

The inner conductor carries a charge density $+\rho_L$ c/m on its surface then equal and opposite charge density $-\rho_L$ c/m exists on the outer conductor.

$$\therefore \; Q = \rho_L \times L \quad \text{---- (1)}$$

Assuming cylindrical co-ordinate system, $\vec{E}$ will be radial from inner to outer and for infinite line charge it is given by

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon r}\,\vec{a_r} \quad \text{--- (2)}$$

$\vec{E}$ is directed from inner conductor to outer conductor. The Potential difference is work done in moving unit charge against $\vec{E}$ i.e. from $r=b$ to $r=a$.

To find Potential difference, consider $\vec{dL}$ in radial direction which is $dr\,\vec{a_r}$.

$$\therefore \vec{dL} = dr\,\vec{a_r} \quad ---- (3)$$

$$\therefore V = -\int_{-}^{+} \vec{E} \cdot \vec{dL}$$

$$= -\int_{-}^{+} \frac{\rho_L}{2\pi\varepsilon r}\vec{a_r} \cdot dr\,\vec{a_r} = -\frac{\rho_L}{2\pi\varepsilon}\left[\ln r\right]_b^a$$

$$= -\frac{\rho_L}{2\pi\varepsilon}\ln\left[\frac{a}{b}\right]$$

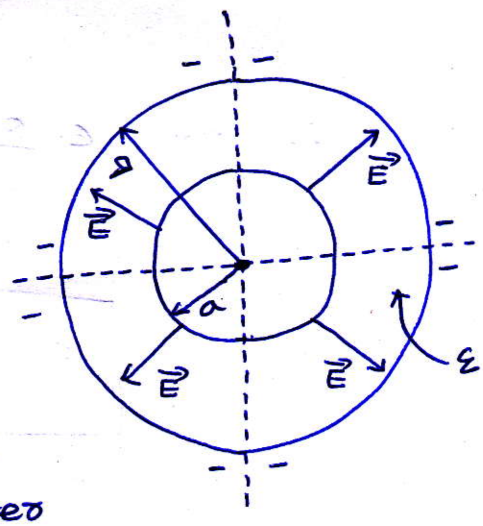
$$\boxed{\therefore V = \frac{\rho_L}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right)}$$

$$\therefore C = \frac{Q}{V} = \frac{\rho_L L}{\frac{\rho_L}{2\pi\varepsilon}\ln\left(\frac{b}{a}\right)} = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}\ F$$

$$\boxed{C = \frac{2\pi\varepsilon L}{\ln\left(\frac{b}{a}\right)}\ F} \quad ---- (4)$$

SPHERICAL CAPACITOR:-

Consider a spherical capacitor formed of two concentric spherical conducting shells of radius 'a' and 'b'. The capacitor is shown in figure.



The radius of outer sphere is 'b' while that of inner sphere is 'a'. Thus $b > a$. The region b/w the two spheres is filled with a dielectric of Permittivity ε.

The inner sphere is given a +ve charge $(+Q)$ while for the outer sphere it is $(-Q)$.

considering, Gaussian surface as a sphere of radius r, it can be obtained that $\vec{E}$ is in radial direction and given by.

$$\vec{E} = \frac{Q}{4\pi\varepsilon r^2}\vec{a_r} \text{ V/m.} \quad ---- (1)$$

[Elementary Equation]

The Potential difference is work done in moving unit positive charge against the direction of $\vec{E}$ i.e from r = b to r = a

$$\therefore \quad V = -\int_{-}^{+} \vec{E}\cdot\vec{dL} = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\vec{a_r}\cdot\vec{dL} \quad --- (2)$$

$$\vec{dL} = dr\,\vec{a_r} \quad --- (3)$$

sub (3) in (2) we get

$$V = -\int_{r=b}^{r=a} \frac{Q}{4\pi\varepsilon r^2}\vec{a_r}\cdot dr\,\vec{a_r}$$

$$= -\int_{b}^{a} \frac{Q}{4\pi\varepsilon r^2}dr = -\frac{Q}{4\pi\varepsilon}\left[-\frac{1}{r}\right]_b^a$$

$$\boxed{V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right] \text{ Volts}} \quad --- (4)$$

$$\therefore \quad \text{Now} \quad C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\varepsilon}\left[\frac{1}{a}-\frac{1}{b}\right]}$$

$$\boxed{C = \frac{4\pi\varepsilon}{\left[\frac{1}{a}-\frac{1}{b}\right]} \text{ F}} \quad --- (5)$$

CAPACITANCE OF SINGLE ISOLATED SPHERE:

Consider a single isolated sphere of radius 'a' given a charge of +Q. It forms a capacitance with an outer plate which is infinitely large hence $b=\infty$.

The capacitance of such a single isolated spherical conductor can be obtained by substituting $b=\infty$ in above eqn. (5)

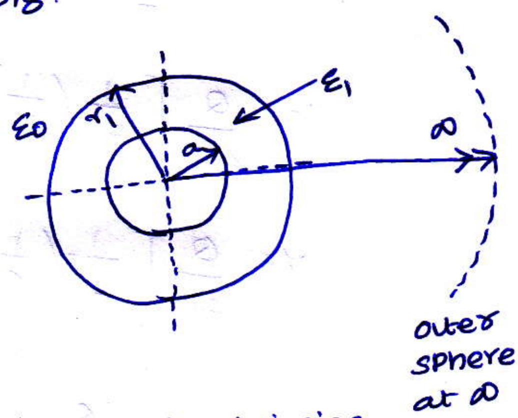
$$\therefore \quad C = \frac{4\pi\varepsilon}{\frac{1}{a}-\frac{1}{\infty}} \quad \text{but } \frac{1}{\infty}=0$$

$$\boxed{C \Rightarrow 4\pi\varepsilon a \text{ farads}}$$

(stray capacitance of an isolated body)

ISOLATED SPHERE COATED WITH DIELECTRIC:

Consider a single isolated sphere coated with a dielectric having permittivity ε_1, upto radius r_1. The radius of inner sphere is 'a' as shown in fig.



It is placed in a free space so outside sphere $\varepsilon=\varepsilon_0$. It carries a charge of +Q.

so for $a<r<r_1$, $\varepsilon=\varepsilon_1$.

for $r>r_1$, $\varepsilon=\varepsilon_0$.

The potential difference is workdone in bringing unit positive charge from outer sphere $r=\infty$ to inner ~~space~~ sphere $r=a$ against $\vec{E}$. This is to be splitted into two as

$$V = -\int_{-}^{+} \vec{E}\cdot\vec{dL} = -\int_{r=\infty}^{r=a} \vec{E}\cdot\vec{dL}$$

$$= -\int_{r=\infty}^{r=r_1} \vec{E}\cdot\vec{dL} - \int_{r=r_1}^{r=a} \vec{E}\cdot\vec{dL} \quad \text{---} (1)$$

NOW for $a < r < r_1$

$$\vec{E}_1 = \frac{Q}{4\pi\varepsilon_1 r^2}\vec{a}_r$$

for $r_1 < r < \infty$

$$\vec{E}_2 = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r$$

while $\vec{dL} = dr\vec{a}_r$

$\therefore$ Eqn ① becomes,

$$V = -\int_{\infty}^{r_1}\frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r \cdot dr\vec{a}_r - \int_{r_1}^{a}\frac{Q}{4\pi\varepsilon_1 r^2}\vec{a}_r \cdot dr\vec{a}_r$$

$$= -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\int_{\infty}^{r_1}\frac{1}{r^2}dr + \frac{1}{\varepsilon_1}\int_{r_1}^{a}\frac{1}{r^2}dr\right]$$

$$= -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left[\frac{-1}{r}\right]_{\infty}^{r_1} + \frac{1}{\varepsilon_1}\left[\frac{-1}{r}\right]_{r_1}^{a}\right]$$

$$V = -\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left(-\frac{1}{r_1} + \frac{1}{\infty}\right) + \frac{1}{\varepsilon_1}\left[-\frac{1}{a} + \frac{1}{r_1}\right]\right]$$

$$V = \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0}\left(\frac{1}{r_1}\right) + \frac{1}{\varepsilon_1}\left(\frac{1}{a}\right) - \frac{1}{\varepsilon_1}\left(\frac{1}{r_1}\right)\right]$$

$$V = \frac{Q}{4\pi}\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}\right] \text{ Volts}$$

$$\therefore \quad C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}\right]}$$

$$C = \frac{4\pi}{\left[\frac{1}{\varepsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}\right]}$$

$$\Rightarrow \frac{1}{C} = \frac{\frac{1}{\varepsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}}{4\pi}$$

$$\Rightarrow \frac{1}{C} = \frac{\frac{1}{a} - \frac{1}{r_1}}{4\pi\varepsilon_1} + \frac{1}{4\pi\varepsilon_0 r_1}$$

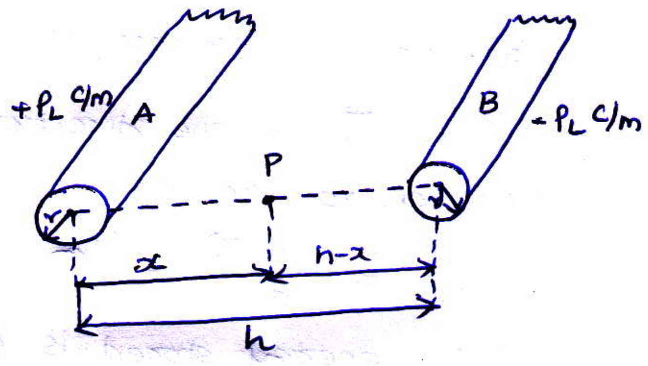
Let $C_1 = \dfrac{4\pi\varepsilon_1}{\frac{1}{a} - \frac{1}{r_1}}$ & $C_2 = 4\pi\varepsilon_0 r_1$

$$\therefore \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\boxed{C = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}}$$

CAPACITANCE BETWEEN TWO TRANSMISSION LINES:

Let us consider two $\parallel^{el}$ conductors A and B of radius 'r' seperated by a distance 'h'. If A has charge $+\rho_L$ c/m along its length, it will induce $-\rho_L$ c/m on conductor, B. At any Point P at a distance x from the centre of A, electric field intensity due to A is.



$$\vec{E}_1 = \frac{\rho_L}{2\pi\varepsilon x}\vec{a}_x$$

Electric field intensity at P due to B is

$$\vec{E}_2 = \frac{-\rho_L}{2\pi\varepsilon(h-x)}(-\vec{a}_x)$$

The Total field Intensity at P is.

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_L}{2\pi\varepsilon}\left[\frac{1}{x} \mp \frac{1}{h-x}\right]\vec{a}_x$$

Potential rise from B to A

$$V = -\int_B^A \vec{E}\cdot\vec{dl}$$

At the surface of A, $x = r$
B, $x = h-r$.

$$\therefore V = -\int_{x=h-r}^{x=r} \frac{\rho_L}{2\pi\varepsilon}\left[\frac{1}{x} \mp \frac{1}{h-x}\right]dx = -\frac{\rho_L}{2\pi\varepsilon}\left[\ln x \mp \ln(h-x)\right]_{h-r}^{r}$$

$$V = \frac{-\rho_L}{2\pi\varepsilon}\left[\ln(r) - \ln(h-r) \mp \ln(h-r) + \ln\left[h-(h-r)\right]\right]$$

$$V = -\frac{\rho_L}{2\pi\varepsilon}\left[2\ln(r) - 2\ln(h-r)\right] = \frac{\rho_L}{\cancel{2}\pi\varepsilon}\cancel{2}\left[\ln(r) - \ln(h-r)\right]$$

$$\therefore V = \frac{\rho_L}{\pi\varepsilon}\left[\ln\left(\frac{h-r}{r}\right)\right]$$

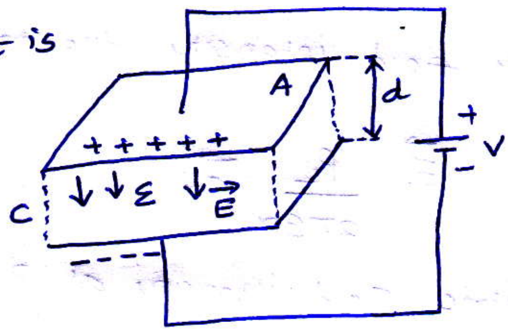
$$\therefore C = \frac{Q}{V} = \frac{\rho_L \cdot l}{\frac{\rho_L}{\pi\varepsilon}\ln\left(\frac{h-r}{r}\right)} = \frac{\pi\varepsilon l}{\ln\left(\frac{h-r}{r}\right)}$$

$$\boxed{\therefore C = \frac{\pi\varepsilon l}{\ln\left(\frac{h-r}{r}\right)}}$$

ENERGY STORED IN A CAPACITOR:

It is seen that capacitor can store the energy. Let's find the expression for the energy stored in a capacitor.

Consider a $\parallel^{el}$ Plate capacitor as shown in the figure. It is supplied with voltage V.



Let $\vec{a}_N$ is the direction normal to the plates

$$\therefore \vec{E} = \frac{V}{d}\vec{a}_N \quad \text{---} ①$$

The energy stored is given by,

$$W_E = \frac{1}{2}\int_{vol} \vec{D}\cdot\vec{E}\,dv$$

$$= \frac{1}{2}\int_{vol} \varepsilon\vec{E}\cdot\vec{E}\,dv \qquad \text{but } \vec{E}\cdot\vec{E} = |\vec{E}|^2$$

$$= \frac{1}{2}\int_{vol} \varepsilon|\vec{E}|^2\,dv \qquad \text{but } |\vec{E}| = \frac{V}{d}$$

$$= \frac{1}{2}\varepsilon\frac{V^2}{d^2}\int_{vol} dv \qquad \text{but } \int_{vol} dv = \text{volume} = A\times d$$

$$= \frac{1}{2}\varepsilon\frac{V^2 Ad}{d^2}$$

$$W_E = \frac{1}{2}\frac{\varepsilon A}{d}V^2 = \frac{1}{2}CV^2 \qquad \left[\because C = \frac{\varepsilon A}{d}\right]$$

ENERGY DENSITY:

Energy density is Energy stored per unit volume as Volume tends to zero.

$$\therefore W_E = \frac{1}{2}\varepsilon\int_{vol} |\vec{E}|^2\,dv$$

$$W_E = \frac{1}{2}\varepsilon|\vec{E}|^2\ J/m^3 = \text{Energy density.}$$

using $|\vec{D}| = \varepsilon|\vec{E}|$ in above expression.

$$W_E = \frac{1}{2}\frac{|\vec{D}|^2}{\varepsilon} = \frac{1}{2}|\vec{D}|\,|\vec{E}|\ J/m^3.$$

POISSON'S & LAPLACE'S EQUATIONS:

From the Gauss's law in the point form, Poisson's equation can be derived.

consider the Gauss's law in the point form as

$$\nabla \cdot \vec{D} = \rho_v \quad \text{---} (1)$$

$\vec{D}$ → Flux density

ρ_v → volume charge density

W.K.T $\vec{D} = \varepsilon \vec{E}$ --- (2)

sub (2) in (1) we get

$$\nabla \cdot \varepsilon \vec{E} = \rho_v \quad \text{---} (3)$$

From the gradient relationship

$$\vec{E} = -\nabla V \quad \text{---} (4)$$

substitute (4) in (3) we get

$$\nabla \cdot \varepsilon(-\nabla V) = \rho_v$$

$$-\varepsilon[\nabla \cdot \nabla V] = \rho_v$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\varepsilon} \quad \text{---} (5)$$

But $\nabla \cdot \nabla = \nabla^2$

$$\therefore (5) \Rightarrow \boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \quad \text{---} (6)$$

Equation (6) is called Poisson's Equation.

If in certain region, $\rho_v = 0$, which is true for dielectric medium then Poisson's Equation takes a form

$$\boxed{\nabla^2 V = 0}$$ (for charge free region).

This is a special case of Poisson's equation called as Laplace's Equation.

∇^2 is called Laplacian of V.

UNIQUENESS THEOREM:-

The boundary value Problems can be solved by number of methods such as analytical, graphical, experimental etc.

Thus there is a question that, is the solution of Laplace's equation solved by any method, unique? The answer to this question is the uniqueness theorem, which is proved by contradiction method.

Assume that the Laplace's equation has two solutions say V_1 and V_2, both are functions of the co-ordinates of the system used. These solutions must satisfy Laplace's equation. So we can write,

$$\nabla^2 V_1 = 0 \quad \text{and} \quad \nabla^2 V_2 = 0 \text{ ---- (1)}$$

Both the solutions must satisfy the boundary conditions as well. At the boundary, the Potentials at different points are same due to equipotential surface then,

$$V_1 = V_2 \text{ --- (2)}$$

Let the difference b/w the two solutions is Vd

$$\therefore \; V_d = V_2 - V_1 \text{ --- (3)}$$

Using Laplace's equation for the difference Vd,

$$\nabla^2 V_d = \nabla^2 (V_2 - V_1) = 0 \text{ ---- (4)} \Rightarrow \nabla^2 V_2 - \nabla^2 V_1 = 0 \text{ --- (5)}$$

on the boundary $V_d = 0$ [from (2) & (3)]

from divergence Theorem,

$$\int_{Vol} (\nabla \cdot \vec{A})\, dv = \oint_S \vec{A} \cdot \vec{ds} \text{ --- (6)}$$

Let $\vec{A} = V_d \nabla V_d$ and from vector algebra

$$\nabla \cdot (\alpha \vec{B}) = \alpha (\nabla \cdot \vec{B}) + \vec{B} \cdot (\nabla \alpha) \text{ --- (7)}$$

Now use this for $\nabla\cdot(V_d \nabla V_d)$ with $\alpha = V_d$ and $\nabla V_d = \vec{B}$

$$\nabla\cdot(V_d \nabla V_d) = V_d(\nabla\cdot\nabla V_d) + \nabla V_d\cdot(\nabla V_d)$$

But $\nabla\cdot\nabla = \nabla^2$ hence

$$\nabla\cdot(V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d\cdot\nabla V_d \quad \text{---} (8)$$

using eqn (4) in (8) ie $\nabla^2 V_d = 0$ we get

$$\nabla\cdot(V_d \nabla V_d) = \nabla V_d\cdot\nabla V_d \quad \text{----} (9)$$

To use this in equation (6)

Let $V_d \nabla V_d = \vec{A}$ hence

$$\nabla\cdot(V_d \nabla V_d) = \nabla\cdot\vec{A} = \nabla V_d\cdot\nabla V_d$$

$$\int_{vol} \nabla V_d\cdot\nabla V_d \, dv = \oint_S V_d \nabla V_d\cdot\vec{ds} \quad \text{---} (10)$$

But $V_d = 0$ on boundary, hence RHS of (10) becomes zero

$$\therefore \int_{vol} (\nabla V_d\cdot\nabla V_d) dv = 0$$

$$\int_{vol} |\nabla V_d|^2 dv = 0 \quad \text{as } \nabla V_d \text{ is a vector} \quad \text{---} (11)$$

Now Integration can be zero under two conditions,

(i) The quantity under integral sign is zero

(ii) the quantity is +ve in some regions and -ve in some other regions by equal amount and hence zero.

$$\therefore |\nabla V_d|^2 = 0$$

$$\nabla V_d = 0.$$

As the gradient of $V_d = V_2 - V_1$ is zero means $V_2 - V_1$ is constant and not changing with any co-ordinates.

But considering boundary it can be proved that

$$V_2 - V_1 = \text{const} = \text{zero}.$$

$$V_2 = V_1$$

This proves that both the solutions are equal and cannot be different.

UNIQUENESS THEOREM states that

If the solution of Laplace's equation satisfies the boundary condition then ~~that~~ solution is unique, by whatever method it is obtained.

TUTORIAL PROBLEMS.

1. verify that the potential field given below satisfies the Laplace's equation $V = 2x^2 - 3y^2 + z^2$

Given field is in cartesian system

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}[2x^2 - 3y^2 + z^2] + \frac{\partial^2}{\partial y^2}[2x^2 - 3y^2 + z^2] + \frac{\partial^2}{\partial z^2}[2x^2 - 3y^2 + z^2]$$

$$= \frac{\partial}{\partial x}[4x] + \frac{\partial}{\partial y}[-6y] + \frac{\partial}{\partial z}[2z]$$

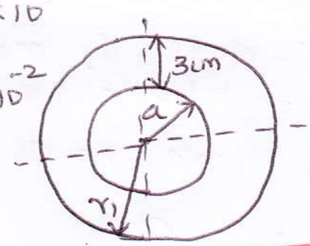
$$\Rightarrow 4 - 6 + 2 = 0$$

$\boxed{\nabla^2 V = 0}$ thus the field satisfies the Laplace's Equation

2. Find the capacitance of a conducting sphere of 2cm in diameter, covered with a layer of polyetheiene with $\varepsilon_r = 2.26$ and 3cm thick.

a = radius of sphere = $\frac{d}{2}$ = 1 cm = 1×10^{-2}

$r_1 = a +$ thickness = 1 + 3 = 4 cm = 4×10^{-2}



$$\therefore C = \frac{4\pi}{\frac{1}{\varepsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\varepsilon_0 r_1}} \quad F$$

$$= \frac{4\pi}{\frac{1}{2.26}\left[\frac{1}{1 \times 10^{-2}} - \frac{1}{4 \times 10^{-2}}\right] + \frac{1}{8.854 \times 10^{-12} \times 4 \times 10^{-2}}}$$

$\boxed{C = 1.9121 \text{ PF}}$

3. A coil of 500 turns is wound on a closed iron ring of mean radius 10 cm and cross section area of 3 cm². find the self inductance of the winding of the relative permeability of iron is 800.

$\mu_r = 800$

$A = 3\text{cm}^2 = 3 \times 10^{-4} \text{ m}^2$

$R = 10\text{ cm} = 10 \times 10^{-2}$ m

$N = 500$

$$L = \frac{\mu N^2 A}{2\pi R}$$

$\mu = \mu_0 \mu_r$

$\mu_0 = 4\pi \times 10^{-7}$

$L = 0.12$ H

$\boxed{L = 120 \text{ mH}}$